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STATISTICAL, NONLINEAR, AND SOFT MATTER PHYSICS

Nonlinear Eigenvalue Problems in Smectics

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Abstract—The asymptotic forms of strains in a smectic around the linear distributions of multipole force are determined. The law of a decrease in strains is specified by the indices, which are eigenvalues of nonlinear equations describing the angular dependence of the strains.

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In some cases, asymptotic forms of strains in smectics are nonlinear effects [1]. Most problems of this type were investigated analytically [2–4]. The situation remained unclear only in one of the problems from the list given in [2], namely, concerning the dipole moment of the force uniformly distributed along the line normal to smectic layers. Here, we will show that in this case, as well as in problems concerning linearly distributed multipole moments, the strains decrease according to a power law, and the exponents in this law can be determined as eigenvalues of the nonlinear equation describing the angular dependence of the strain. At a certain macroscopic distance, displacements become on the order of the smectic period. Upon a further increase in the radius, a universal behavior is observed for a dipole, while a logarithmic law of the decrease operates for a quadrupole; for higher moments, a linear approximation is applicable.

The elastic strain energy of a smectic with small deviations of the orientation of layers from the horizontal (xy) plane is given by ([5], expressions (44), (13))

$$E = \frac{A}{2}\int\left(\left(\partial_z u - \frac{(\partial_\mu u)^2}{2}\right)^2 + (\lambda\Delta_\perp u)^2\right)dV, \qquad (1)$$

where $u(\mathbf{r})$ is the displacement of layers along the vertical (z axis), $A > 0$ is the elastic constant, λ is a quantity on the order of the smectic period, and

$$\Delta_\perp = \partial_x^2 + \partial_y^2, \quad (\partial_\mu u)^2 = (\partial_x u)^2 + (\partial_y u)^2.$$

In the given problem, which is homogeneous along the z axis, the vertical distance between the smectic layers remains unchanged. Therefore, $\partial_z u = 0$ as in the initial (prior to the application of forces) undeformed state. Then the equilibrium equation acquires the form

$$\lambda^2\Delta_\perp^2 u - \frac{1}{2}\partial_\nu\{(\partial_\mu u)^2\partial_\nu u\} = \frac{\mathcal{F}(x, y)}{A}, \qquad (2)$$

where $\mathcal{F}$ describes the external action on the smectic.

Let us first consider the limit of large displacements $u \gg \lambda$, disregarding the first term in Eq. (2). It should be noted that when a linearly distributed force is applied, the strain decreases in accordance with the law $\partial_\rho u = -\rho^{-1/3}$ [3]. Here, the exponent can be determined from the requirement that the applied force be equal to the integral of stress $\sigma_{z\rho} \propto (\partial_\rho u)^3$ over a cylindrical surface of radius ρ. An analogous procedure for the linear distribution of the dipole moment of force proves incorrect. The independence of the moment of forces acting on the arbitrary cylindrical surface of the radius leads to the law $u \propto \rho^{-1/3}$. However, Eq. (2) has no solution of the form $u = f(\theta)\rho^{-1/3}$, where f is a periodic function of the azimuth.

To resolve this situation, paradoxical from the standpoint of the momentum conservation law, we solved the problem numerically. The moment of force is applied with the help of a pair of identical and oppositely directed linear distributions of forces:

$$\mathcal{F}(\mathbf{r}) = F\delta(y)(\delta(x-d) - \delta(x+d)).$$

We assume that the smectic layers are fixed ($u = 0$) at the walls of a vertical cylinder of radius $R \gg d$.

Figure 1 shows the results of numerical solution of the dipole problem. At distances $d \ll \rho \ll R$, the power law

$$u = \left(\frac{Fd^2}{A}\right)^{1/3}\left(\frac{\rho}{d}\right)^\alpha f(\theta) \qquad (3)$$

holds, where $\alpha \approx -0.476$. Angle θ will be measured from the x axis. In fact, asymptotic form (3) is observed for $5d < \rho < 0.2R$.

Thus, the moment of forces acting on the cylindrical surface decreases with increasing distance from the cylinder axis. In this case, we must consider the effects at the ends of an arbitrarily selected cylinder of a finite height. In the nonlinear case considered here, the anisotropic part of the stress tensor in the smectics is $\sigma_{ik} = An_in_k\delta a/a$, where $\delta a/a$ is the relative change in the interplanar distance in the smectics and n_i are the components of the unit vector normal to the smectic layers. If $\partial_z u = 0$, for small angles we have

$$\sigma_{xz} = A\partial_x u \frac{\delta a}{a} = -\frac{A}{2}\partial_x u(\partial_\mu u)^2.$$

The momentum conservation law is satisfied if we take into account the moment of forces

$$LF = L\int \sigma_{xz} dS$$

acting at the ends of the cylinder of height L. This is confirmed in the numerical solution. The forces acting on the upper and lower end faces are of the same magnitude and of opposite directions, and they are mainly accumulated over distances on the order of d.

The numerical results described above considerably simplify the analytic calculation of nonlinear multipole strains.

Dipole, $\lambda = 0$. The fact that the strain amplitude u in a nonlinear regime obeys the law $u \propto (F/A)^{1/3}$ follows from equilibrium equation (2). Function f and exponent α can be determined as the eigenfunctions and eigenvalues of the following nonlinear problem. We seek the solution to Eq. (2) in the form (3). For function f, we obtain the equation

$$(\alpha^2 f^2 + 3\dot{f}^2)\ddot{f} + \alpha(5\alpha + 2)f\dot{f}^2 + \alpha^3(3\alpha - 2)f^3 = 0, \quad (4)$$

in which the dot indicates differentiation with respect to angle θ. This equation has a single-valued (i.e., periodic) solution $f(\theta) = f(\theta + 2\pi)$ only for a discrete set of values of parameter α. The order of the equation can be lowered if we seek f as a function of a new variable $\eta = \dot{f}/f$. Then the first integral to Eq. (4) can be found easily and can be written in the form

$$f^2(\eta^2 + \alpha^2)^\alpha\left(\eta^2 + \alpha^2 - \frac{2}{3}\alpha\right)^{1-\alpha} = C, \quad (5)$$

where C is the integration constant. For angle θ, we have $\theta = \int (f\eta)^{-1} df$. Using relation (5), we obtain

$$\theta = -\arctan\frac{\eta}{\alpha} - \frac{(1-\alpha)}{\sqrt{(\alpha - 2/3)\alpha}} \times \arctan\frac{\eta}{\sqrt{(\alpha - 2/3)\alpha}}. \quad (6)$$

The integration constant is zero because the maximum of function f is obviously attained for $\theta = 0$. In this case, $\eta \propto \dot{f} = 0$. It follows from the symmetry of the dipole distribution of forces that the sign reversal for function f must take place for $\theta = \pm\pi/2$. In accordance with relation (5), $\eta \longrightarrow \mp\infty$ in this case. Equation (6) can then be reduced to

$$\frac{\pi}{2} = -\frac{\pi}{2} + \frac{1-\alpha}{\sqrt{(\alpha - 2/3)\alpha}}\frac{\pi}{2} \quad (7)$$

(condition $\theta \longrightarrow +\pi/2$ corresponds to $\eta \longrightarrow -\infty$). This gives a value of $\alpha = (1 - 2\sqrt{7})/9$.

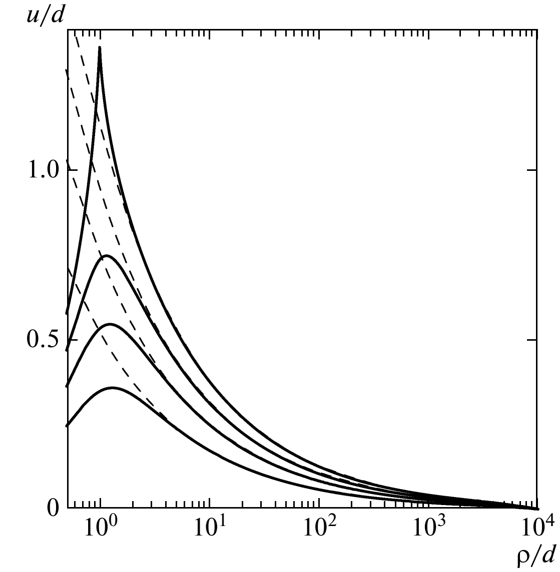

Fig. 1. Field of a dipole for $\lambda = 0$ as a function of the radius for $F/Ad = 1$ at angles $\theta = 0, \pi/6, \pi/4$, and $\pi/3$. Solid curves describe the numerical solution, and dashed curves are asymptotic forms (3).

Multipoles, $\lambda = 0$. Solution (5), (6) is a single-valued function of angle θ for an infinite discrete set of indices

$$\alpha = \frac{-2n^2 + 2n + 1 - (n+1)\sqrt{4n^2 + 2n + 1}}{3(2n+1)}, \quad (8)$$

for which the angular distance between the nearest, say, maxima of function f is $2\pi/n$ (where $n = 2m$). Such fields correspond to $2m$-pole distributions of forces along a line parallel to the symmetry axis of the smectic.

The resultant nonlinear asymptotic forms are correct as long as the displacement amplitude considerably exceeds λ. This condition is violated at distances

$$\rho \sim \rho^* \sim d(Fd^2/A\lambda^3)^{1/3|\alpha|};$$

in this case, the first term in equilibrium equation (2) must be taken into account. In addition, one more boundary condition should be imposed. For such a condition, we choose $\Delta_\perp u = 0$ for $\rho = R$. This is a natural variational condition in the case of free orientation of smectic layers at the vessel walls.

Figure 2 shows the results of numerical solution of problems for the dipole, quadrupole, and next-order (hexapole) moments for the general ($\lambda \neq 0$) equation (2). It can be seen that under a strong action, two macroscopic regions exhibiting different behaviors can be distinguished. At distances $d \ll \rho \ll \rho^*$, the asymptotic form (3) considered above is observed. At distances $\rho^* \ll \rho \ll R$, the far asymptotic form is realized. In this case, the displacements behave differently in the dipole, quadrupole, and hexapole: the displacements are independent of radius for a dipole (Fig. 2a), slowly

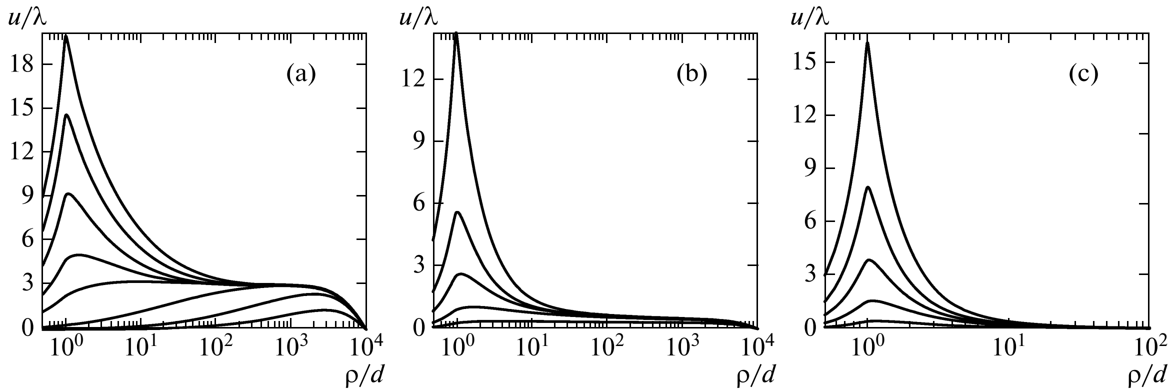

Fig. 2. Evolution of the field of displacement u as a function of the radius for $\theta = 0$ upon an increase in applied forces: (a) dipole, $F/Ad = 0.005$, 0.038, 20, 60, 100, 500, 1700, and 4000; (b) quadrupole; $F/Ad = 1$, 5, 20, 100, 500, and 5000; (c) hexapole; $F/Ad =$ 20, 100, 500, 2500, and 15000.

decrease in the case of a quadrupole (Fig. 2b), and rapidly decrease in the case of a hexapole (Fig. 2c) and higher-order multipoles.

For multipoles beginning with a hexapole, the amplitude of displacement at distances $\rho \gg \rho^*$ is much smaller than λ. In this case, nonlinear effects can be ignored, and the standard multipole asymptotic forms ($u \propto \cos(m\theta)\rho^{-m+2}$, where $m \geq 3$) of the linear equation $\Delta_\perp^2 u = 0$ hold.

However, in the case of a dipole and a quadrupole, the nonlinearity is significant at any distance.

Dipole, $\lambda \neq 0$. The behavior of the displacement field at distances $\rho^* \ll \rho \ll R$ observed in numerical calculations (see Fig. 2a) indicates the possibility that a solution $u = \lambda f(\theta)$ exists, which is independent of ρ. For function f, we obtain the equation

$$\ddddot{f} + 4\ddot{f} - \frac{3}{2}\dot{f}^2\ddot{f} = 0, \tag{9}$$

which can be integrated in quadratures. The first integral is $\dddot{f} + 4\dot{f} - \dot{f}^3/2 = C_1$. Due to the symmetry of the problem, integration constant C_1 turns to zero. The second integral is

$$\frac{\ddot{f}^2}{2} + 2\dot{f}^2 - \frac{\dot{f}^4}{8} = C. \tag{10}$$

Let us introduce an auxiliary variable φ such that $\dot{f} = -b\sin\varphi$, where b is the maximal value of $\dot{f}$. Relation (10) implies that $C = 2b^2 - b^4/8$. In this case, the second integral can be written in the form

$$\dot{\varphi}^2 = 4 - \frac{b^2}{4} - \frac{b^2}{4}\sin^2\varphi. \tag{11}$$

This gives the third integral of the problem,

$$\theta = \frac{2}{\sqrt{16 - b^2}}F(\varphi, k), \tag{12}$$

where $F(\varphi, k)$ is the elliptical integral

$$\int_0^\varphi (1 - k^2\sin^2\varphi)^{-1/2}d\varphi$$

and $k = b/\sqrt{16 - b^2}$. Angle θ is measured from the point of maximum of functions f ($\varphi = 0$ in this case). Finally, the fourth integral is given by

$$f = -b\int\sin\varphi d\theta = \ln\frac{\sqrt{1 - k^2\sin^2\varphi} + k\cos\varphi}{\sqrt{1 - k^2\sin^2\varphi} - k\cos\varphi}. \tag{13}$$

Here, the integration constant is set to zero in accordance with the symmetry of the field of displacements caused by the dipole distribution of the force.

Function f changes its sign for $\varphi = \pi/2$. For a dipole, this must take place for $\theta = \pi/2$. Accordingly, parameter b is defined by the relation

$$\frac{\pi}{2} = \frac{2}{\sqrt{16 - b^2}}F\left(\frac{\pi}{2}, k\right), \tag{14}$$

$b \approx 2.691$.

It should be noted that eigenfunction $f(\theta)$ of a distant ($\rho \gg \rho^*$) asymptotic form, as in the case of a near ($\rho \ll \rho^*$) asymptotic form (3), is mainly (by approximately 95%) determined by the first angular harmonic of $\cos\theta$.

We pay attention to the fact that the solution obtained here is not only universal (i.e., independent of the intensity of action; its realization requires only a rather strong force), but it also independent of the dipole moment.

Quadrupole, $\lambda \neq 0$. Note that solutions independent of radius, which could correspond to higher-order multipoles, do not exist. As a matter of fact, the minimal value of the right-hand side of dispersion relation (14) is $\pi/4$. It is attained for $b \longrightarrow 0$, i.e., for the amplitude of the solution tending to zero. Therefore, the value $\pi/4$ satisfying the quadrupole is unattainable. This fact, as well as the form of the numerical solution (see Fig. 2b) at distances $\rho^* \ll \rho \ll R$, indicates the possibility of a solution in the form

$$u = \lambda f(\theta)\ln^{\alpha}\frac{\rho}{\rho^*}. \quad (15)$$

For $\alpha < 0$ (which can naturally be assumed because we have $\alpha = 0$ for a dipole moment), in the main approximation, only the linear term

$$(\ddddot{f} + 4\ddot{f})\frac{\lambda^3}{\rho^4}\ln^{\alpha}\frac{\rho}{\rho^*} \quad (16)$$

is left in Eq. (2). Thus, $f = C\cos 2\theta$, where C is a certain constant. The next-approximation terms in the linear term of the equilibrium equation are given by

$$-8\alpha C\cos 2\theta\frac{\lambda^3}{\rho^4}\ln^{\alpha-1}\frac{\rho}{\rho^*}, \quad (17)$$

while in the nonlinear term, we have

$$-3C^3(\cos 2\theta - \cos 6\theta)\frac{\lambda^3}{\rho^4}\ln^{3\alpha}\frac{\rho}{\rho^*}. \quad (18)$$

Canceling out the terms proportional to $\cos 2\theta$, we obtain exponent $\alpha = -1/2$ and constant $C = 2/\sqrt{3}$. To compensate the term proportional to $\cos 6\theta$, we must take into account in expression (18) a correction proportional to $\lambda\cos 6\theta\ln^{-3/2}(\rho/\rho^*)$ to function (15). Thus, the quadrupole solution at distances $\rho \gg \rho^*$ can be written in the form of a series in reciprocal half-integer powers of the logarithm.

At distances $\rho \sim R$, the asymptotic forms described above are violated for all multipoles. Here, the solutions are matched to the boundary conditions at the vessel walls.

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