Localization in one-dimensional incommensurate systems

I. M. Suslov

Institute of Physics Problems, USSR Academy of Sciences
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The motion of electrons in the field of two one-dimensional periodic potentials with incommensurate periods is considered. The existence of an Anderson transition in such systems is demonstrated. The critical exponent of the wave-function localization radius is calculated and found to be unity.

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INTRODUCTION

In connection with recent reports of new experimental procedures, it has become possible to produce the so-called incommensurate systems (see Ref. 1 and the literature cited therein). The simplest example of such a system is a monatomic film with period a, located on a crystal substrate with period b, so that the ratio $\beta = a/b$ is an irrational number (the case of a rational number ratio is of no interest, since the system is then periodic with a larger period). We are interested in the case of a metallic film on a dielectric substrate. In connection with the electronic properties of such systems, the naturally raised question is that of the motion of an electron in the field of two periodic potentials with incommensurate periods.

These systems are of interest because they constitute an intermediate case between periodic and disordered systems—they have no translational symmetry, but likewise no real disorder; in particular, Anderson localization, which has attracted much attention of late, is possible in them.[2,3] Arguments favoring the existence of localization in incommensurate systems are advanced by Azbel,[4] Aubry,[5] and Sokoloff *et al.*,[6] as well as the results of numerical experiments.[7,8] We show in the present paper that for a definite class of incommensurate systems it is possible to construct a consistent localization theory, and calculate the mobility threshold and the critical exponent of the localization radius.

Since the Anderson transition has been observed already in incommensurate system in the one-dimensional case, we confine ourselves to this case, which also has a bearing on systems such as mercury chains (see the experimental studies cited in Ref. 4). We note that at the present time we do not know of any other model that admits of an analytic solution in which the Anderson transition exists (including the Bethe solution).

1. LOCALIZATION IN THE SIMPLEST MODEL

The simplest model that describes the situation under consideration is the tight-binding model with periodic modulation of the levels:

$$a_{l+1}+a_{l-1}+V\cos(2\pi\beta l)\,a_l = Ea_l, \tag{1}$$

where β is the irrational incommensurability parameter (the energy is measured in units of the overlap integral J). Without loss of generality we can assume that $\beta < 1$ (a change of β by an integer does not change the Hamiltonian).

As shown by Azbel,[4] the classification of the states of an incommensurate system is determined by expansion of β into a continued fraction:

$$\beta = \frac{1}{n_1+\beta_1} = \frac{1}{n_1+\dfrac{1}{n_2+\beta_2}} = \ldots = \frac{1}{n_1+\dfrac{1}{n_2+\dfrac{1}{n_3+\ldots}}}. \tag{2}$$

The reason why the continued fraction (2) appears in the theory can be easily understood. An irrational number β can be naturally regarded as a limit of a sequence of rational numbers. From all these sequences we single out the sequence of rational numbers p_n/q_n obtained by terminating the continued fraction (2) at the n-th step. These numbers (and only these) have a definite extremal property—each of the gives, in a certain sense, the best estimate of β (namely, p_n/q_n has the minimum value of the quantity $|q_n\beta - p_n|$ from among all the fractions with a denominator does not exceed q_n (Ref. 9)). This sequence of rational numbers generates a sequence of periodic systems that approximate in extremal manner the given incommensurate one. The periods of these systems (the interatomic distance is $L_0 = 1$)

$$L_1 \sim \frac{1}{\beta}, \quad L_2 \sim \frac{1}{\beta\beta_1}, \quad \ldots, L_n \sim \frac{1}{\beta\beta_1\ldots\beta_{n-1}}, \ldots \tag{3}$$

are the characteristic lengths of the considered incommensurate system and manifest themselves in one manner or another in all the observed phenomena.

To obtain a lucid physical picture, we consider a situation in which there is an hierarchy of lengths (3), i.e., when $\beta, \beta_1, \beta_2 \ldots \ll 1$. By the same token, we restrict the analysis to irrational numbers of a definite type.[1)] The qualitative picture for arbitrary irrational numbers will be presented in Sec. 3.

The problem consists of bringing to light the character of the solutions of Eq. (1). At $V = 1$, its spectrum is a band of width ~ 1. It is clear that this band is preserved in some form also at finite V. Considering the edge of the band, we can replace, by virtue of the condition $\beta \ll 1$, the difference operator by a differential one (since a_l will be a slowly varying function). We then obtain an ordinary Schrödinger equation with a periodic potential

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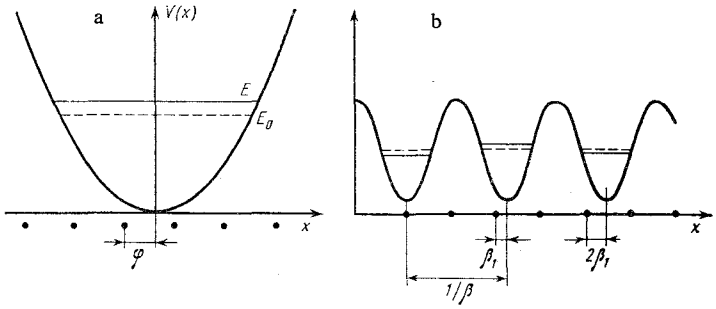

FIG. 1. a) The level of the discrete Schrödinger equation, E, oscillates as a function of the phase φ that determines the position of the discrete system of points on which the equation is specified. b) The phase φ is different for different minima of the periodic potential.

$$\left[\frac{d^2}{dx^2}+V\cos(2\pi\beta x)\right]a(x)=Ea(x), \tag{4}$$

whose spectrum consists of an aggregate of bands with exponentially small width (relative to $\sim\beta$), the distance between them is $\sim\beta$. Thus, in this approximation the initial band of Eq. (1) turns out to be split into a large number ($\sim 1/\beta$) of narrow bands (we shall call them first-order bands), and the wave functions are Bloch waves.

By going from (1) to (4), however, we have completely lost the incommensurability: in Eq. (1) there are two periods, 1 and $1/\beta$, whereas in (4) there is only one, $1/\beta$. It turns out, however, that Eq. (4) gives a correct first-order approximation, but it is necessary to take into account also the principal discreteness effect.

To bring to light the discreteness effect, we consider a difference equation of the type (1), but with a potential in the form of one well [Fig. 1(a)] in place of the periodic potential $V\cos(2\pi\beta x)$. Assume that in the continuous approximation (as the interval of the difference operator tends to zero) a level E_0 exists in the well. Then we have also for the discrete equation a level E, at a small interval (or a slowly varying potential) close to E. However, the discreteness leads to a new effect—to a dependence of the level E on the phase φ [Fig. 1(a)], which determines the position of the discrete system of points singled out by the difference operator, relative to bottom of the well. Since a change of φ by unity changes nothing, this dependence will be periodic. For a slowly varying potential this dependence is described by one harmonic (see Sec. 2):

$$E(\varphi)-E_0\propto\cos 2\pi\varphi. \tag{5}$$

We turn now to the spectrum of Eq. (1) and consider one of the first-order bands. This band was the result of broadening of the level corresponding to motion in one of the minima of the potential, owing to the tunneling into the next minima (the smallness of β ensures applicability of the tight-binding approximation). In the continuous approximation corresponding to Eq. (4), the levels are the same in all the wells; when the discreteness is taken into account, they turn out to be different, since the phase φ is different for the different wells [Fig. 1(b)]: for the k-th well $\varphi_k=\beta_1 k$, where $\beta_1=1/\beta-[1/\beta]$ [i.e., it coincides with β_1 from (2)], so that the level E_k changes as a result of (5) in proportion to $\cos(2\pi\beta_1 k)$. Taking into account the overlap of the wave functions corresponding to motion in individual wells (i.e., using the discrete analog of the tight-binding approximation) and confining ourselves to nearest neighbors, we find that each of the first-order bands is described by an equation of type (1) with β_1 in place of β. Consequently, each of the first-order bands is arranged in the same manner as the initial band, i.e., it turns out to be split up into a large number ($\sim 1/\beta_1$) of bands of second order etc.—the so-called devil's stairs is produced.[4]

The foregoing points clearly to a procedure for constructing the wave functions of Eq. (1). We initially have Wannier functions at the lattice site, with a period $L_0=1$, whose amplitudes are described by Eq. (1). In first-order approximation, i.e., on the first level of the devil's stairs, we should form new Wannier functions (of first order) located at the minima of the potential $\cos(2\pi\beta x)$, i.e., at distances $L_1\sim 1/\beta$ from one another (Fig. 2a), and with amplitudes described by an equation of type (1) but with coefficients β_1 (from (2)) in place of β and with V_1 in place of V. Analogously, on the n-th level of the devil's stairs we obtain from the Wannier functions of order $(n-1)$ the Wannier functions of n-th order with distance $L_n\sim 1/\beta\beta_1\ldots\beta_{n-1}$ between them, and with amplitudes described by Eq. (1) in which β and V is replaced by β_n and V_n. It is required to determine what is obtained in the limit as $n\to\infty$.

We note that the limit $n\to\infty$ can formally be ascribed a physical meaning: it corresponds to the system length L tending to infinity (since the levels to which lengths $L_n>L$ correspond exert no influence on the properties of the system). Let $L\sim L_n$; then the sample length accommodates one Wannier function of order n (Fig. 3). Since the Wannier func-

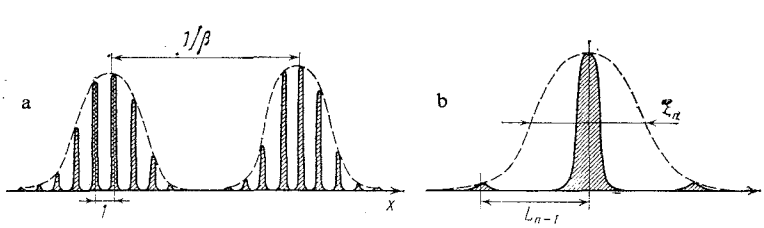

FIG. 2. a) Formation of first-order Wannier functions from the initial Wannier functions. b) Cessation of modifications of the Wannier functions at $V_n\gtrsim 1/\beta_n$.

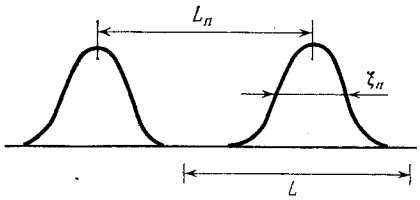

FIG. 3.

tion decreases exponentially on both sides, we have a certain analog of localization; this, however, is not a true localization, since the radius of localization of the wave function ξ_n is of the same order as the dimension L of the system. We increase the system dimension to $\sim L_{n+1}$, and then the sample length will accommodate one Wannier function of order $(n+1)$ with localization radius ξ_{n+1}, etc. Obviously, for localization to exist we must have $\xi_n/L_n \to 0$ as $n \to \infty$. To this end it is necessary in turn that in Eq. (1) the sequence of the coefficients $V_n \to \infty$ as $n \to \infty$.

It is easily seen that localization does indeed exist at a sufficiently rapid growth of V_n. After V_n reaches a value $\sim 1/\beta_n$ (in units of the overlap integral J_n), the localization radius ξ_n of the Wannier function of n-th order becomes smaller than the distance L_{n-1} between the Wannier function of order $(n-1)$, and modification of the Wannier functions no longer takes place [see Fig. 2(b)]—the Wannier functions of order n and $n-1$ turn out to coincide physically. By the same token the localization radius ξ_n takes on a constant value ξ.

Thus, the problem reduces to finding the law for the transformation of the coefficient V_n:

$$V_{n+1}=F_n(V_n) \tag{6}$$

and to a determination of the behavior of V_n as $n \to \infty$.

It will be shown in 2 that the transformation (6) has the following properties:

$$F_n(2)=2, \tag{7a}$$

$$V<V_1<V_2<\ldots \quad \text{at} \quad V>2, \tag{7b}$$

$$V_{n+1}\sim V_n^{1/\beta_n} \quad \text{at} \quad V_n \gg 1, \tag{7c}$$

$$V>V_1>V_2>\ldots \quad \text{at} \quad V<2, \tag{7d}$$

$$V_{n+1}\sim V_n^{1/\beta_n} \quad \text{at} \quad V_n \ll 1. \tag{7e}$$

Thus, at $V>2$ the sequence V_n is an increasing function and has at $n \to \infty$ the following growth order (at $\beta \sim \beta_1 \sim \beta_2 \sim \ldots$):

$$\ln \ln V_n \sim n. \tag{8}$$

On the contrary, at $V<2$ they decrease at the same rate. Obviously, $V=2$ is the Anderson transition point: at $V>2$ all the states of the considered incommensurate system are localized, and at $V<2$ all the states are delocalized. There are no mobility energy thresholds in this model. This result agrees with the data of the numerical experiments.[7]

We ascertain now the behavior of the localization radius near the Anderson transition. Let N be the number of the devil's-stairs level on which the Wannier function ceases to become modified. Then, obviously,

$$\xi \sim \xi_N \sim L_N \sim \frac{1}{\beta\beta_1 \ldots \beta_{N-1}}. \tag{9}$$

It remains to determine N, i.e., to find out after how many steps the sequence V_n increase from the value $V_0 = V \approx 2$ to the value $V_N \sim 1/\beta_N$, which can be regarded, by virtue of the rapid growth of V_n, as independent of N. Being interested in the value of ξ accurate to a constant factor, we can determine n accurate to an additive constant. We can therefore choose V_N to be any constant: it is convenient to choose it close to 2. Expanding (6) near the transition, we obtain the equation

$$(V_{n+1}-2)=A_n(V_n-2), \quad V_0=V,$$

whose solution is

$$V_N-2=A_0A_1\ldots A_{N-1}(V-2). \tag{10}$$

It remains to put $V_n = \text{const}$ and eliminate N from (9) and (10). Calculation of the coefficients A_n in the tight-binding approximation yields $A_n = 1/\beta_n$ (see Sec. 2). We therefore obtain from (9) and (10)

$$\xi \propto (V-2)^{-1}, \tag{11}$$

i.e., the localization radius diverges near the transition in accord with a power law, with a universal critical exponent equal to unity. It is curious that according to the numerical calculations for the three-dimensional Anderson model[10] this critical exponent is equal to 1.2 ± 0.3, i.e., it turns out to be the same within the limits of error.

We have obtained essentially single-parameter scaling close to that proposed in Ref. 11. In particular, at $\beta=\beta_1=\beta_2=\ldots$ the transformation (6) does not depend on n and can be written in the form of the "renormalization-group equation"

$$\Delta \ln V_n/\Delta \ln L_n=\bar{\beta}(V_n), \tag{12}$$

where $\Delta f_n \equiv f_{n+1}-f_n$ and it is recognized that $L_n \sim 1/\beta^n$, i.e., $n \sim \ln L_n$. The "Gell-Mann-Low" function $\beta(V_n)$ increases monotonically (with increasing V_n) from $-\infty$ to $+\infty$ ($\bar{\beta}(V_n) \sim \ln V_n$ at $V_n \gg 1$ and $V_n \ll 1$), and goes through zero at $V_n = 2$.

2. CONVERSION OF THE COEFFICIENTS V_n ON THE DEVIL'S STAIRS

We show first how to calculate the effect of the oscillations of the discrete Schrödinger equation as a function of the phase φ [see Fig. 1(a)]. For the sake of clarity, we start with an analysis of a discrete oscillator:

$$a_{l+1}+a_{l-1}+(l+\varphi)^2 a_l=Ea_l. \tag{13}$$

We shall assume Eq. (13) to be specified not on a discrete set of points $x = 1, \pm 1, \ldots$, but on the entire x axis. With the aid of the shift operators $e^{i\hat{p}}$ ($\hat{p} = -id/dx$) we reduce it to the form

$$[2\cos \hat{p}+x^2]\,a(x)=Ea(x) \tag{14}$$

(the phase φ is made to vanish by the substitution $x+\varphi \to x$).

Changing over to the momentum representation ($\hat{p}\to p$, $x\to\dot{x}=id/dp$), we obtain a Schrödinger equation with a periodic potential

$$[-d^2/dp^2+2\cos p]\,a(p)=Ea(p), \tag{15}$$

whose spectrum is an aggregate of bands. In place of the levels of discrete oscillator we obtain bands, since Eq. (14) has more solutions than (13)—the information on the phase φ is lost in (14). The solutions of Eq. (13) are picked out of the solutions of (14) with the aid of the boundary conditions (in the momentum representation)

$$a(p+2\pi)=e^{i2\pi\varphi}a(p). \tag{16}$$

Indeed, at $\varphi=0$ the function $a(x)$ differs from zero only at integer points, and its Fourier transform is periodic with a period 2π; the shift of the discrete system of points by φ leads to the appearance in (16) of a phase factor. According to (16), the phase φ plays for Eq. (15) the role of a quasimomentum: at a fixed φ there is selected from each band of (15) a level corresponding to the quasi-momentum φ; when φ changes from zero to unity the selected level runs through the entire band and returns to the initial position. Thus, the s-th level of the discrete oscillator varies like

$$E_s(\varphi)=\text{const}+\varepsilon_s(\varphi),$$

where $\varepsilon_s(q)$ is the dispersion law of the s-th band of Eq. (14). For the quasiclassical potential, the spectrum of Eq. (15) is described by the tight-binding approximation, and the dispersion law $\varepsilon_s(q)$ takes the form of a simple cosine—this proves Eq. (5).

In essence, we did not use in the foregoing reasoning any specific properties of the potential x^2. In the case of an arbitrary potential $V(x)$ we arrive at Eq. (15), but with a more general $V(x)$ dispersion law in place of $x^2=-d^2/dp^2$.

We proceed to derive the transformation law (16). For the sake of argument we consider the first level of the devil's stairs. Equation (1) is equivalent to the equation

$$[\cos\hat{p}+{}^1/_2V\cos(2\pi\beta x)]\,a(x)=Ea(x)$$

with boundary condition (16). Writing for it the quasiclassical-quantization conditions, we obtain $\sim 1/\beta$ levels E_s corresponding to motion in one of the minima of the potential (there is no modulation of the levels in this approximation). Allowance for the small probability of tunneling into neighboring minima leads to broadening of these levels into bands of width

$$\Gamma_s^A=\Gamma_s(V/2)\equiv\frac{2\omega}{\pi}\exp\left\{-\int|p|\,dx\right\},$$
$$|p(x)|=\operatorname{arch}\{{}^1/_2V\cos(2\pi\beta x)-E_s\} \tag{17}$$

(see problem 3 of Sec. 55 in Ref. 12); here ω is the frequency of the classical oscillations and the integration in the argument of the exponential is over the classically inaccessible region. The band width Γ_s^A determines the overlap integral J_1 on the next level of the devil's stairs ($\Gamma_s^A=4J_1$). As is clear from the statements made at the beginning of the section, the amplitude V_1 of the level modulation is determined by the band width Γ_s^B of the equation conjugate to (A) ($x\to\hat{p}$, $\hat{p}\to x$)

$$[{}^1/_2V\cos(2\pi\beta\hat{p})+\cos x]\,a(x)=Ea(x),$$

which, following the substitutions $x=2\pi\beta\tilde{x}$ and $E\equiv{}^1/_2V\tilde{E}$, reduces to the form (A) with $2/V$ in place of $V/2$, so that

$$\Gamma_s^B={}^1/_2V\Gamma_s(2/V).$$

Expressing $V_1=\Gamma_s^B/2$ in units of the overlap integral $J_1=\Gamma_s^A/4$, we obtain

$$V_1^{(s)}=V\frac{\Gamma_s(2/V)}{\Gamma_s(V/2)}, \tag{18}$$

i.e., the transformation (6) with $n=0$ it has the same form also in the succeeding levels.

It is easy to present a formal derivation of (18). We neglect in (2) the small quantity β_1; then the incommensurate system is transformed into a periodic one having a period of n_1 atoms. Therefore, the equation (A) with the boundary conditions (16) will have a spectrum of n_1 bands with dispersion laws

$$\varepsilon_s^A(p,\varphi),\quad 1\leqslant s\leqslant n_1,$$

that depend on φ as a parameter. In view of the periodicity of the system, the phase φ will be the same for all the periods of the potential of Eq. (A), so that the Schrödinger equation for the s-th band takes the form

$$\varepsilon_s^A(\hat{p},\varphi)\,a(x)=Ea(x). \tag{19}$$

At finite but small β_1, the phase φ becomes slowly varying in space: $\varphi=\beta_1x$ [see Fig. 1(b)], but Eq. (19) is preserved in first-order approximation. Since the phase φ plays the role of a quasimomentum for the conjugate equation (B), and the quasimomentum p plays the role of the phase, we have obviously

$$\varepsilon_s^A(p,\varphi)=\varepsilon_s^B(\varphi,p), \tag{20}$$

where $\varepsilon_s^B(p,\varphi)$ is the dispersion law of the s-th band of Eq. (B). If the tight-binding approximation is applicable to Eqs. (A) and (B), the quantities ε_s^A and ε_s^B have a cosinusoidal variation in p. Noting also that the modulation of the overlap integral is small compared with the modulation of the levels, we find from (20) that

$$\varepsilon_s^A(p,\varphi)=\text{const}+{}^1/_2\Gamma_s^A\cos p+{}^1/_2\Gamma_s^B\cos 2\pi\varphi.$$

Substituting this in (19) with $\varphi=\beta_1x$, we obtain an equation of the same form as (A) with $V_1=2\Gamma_s^B/\Gamma_s^A$ in place of V, which is equivalent to (18).

The properties (7) of the transformation (6) are derived without almost any calculations. The property (7a) is obvious from (18). The properties (7b) and (7d) follow from the fact that the widths of the allowed bands of the Schrödinger equation decrease with increasing potential, and in the quasiclassical approach this dependence is very strong. Finally, to establish the property (7c) [and analogously (7e)], we note that the ratio of the width of the s-th band of Eq. (A) to the width of the s-th band of Eq. (B) is determined at $V\gg 1$ by an exponentially small transmission coefficient D through the barrier separating the two minima of the potential $V\cos(2\pi\beta x)$; in the case considered, $D\sim V^{-\text{const}/\beta}$. Calculation of the coefficient A_n in (11) is carried out using Eqs. (17) and (18) and raises no difficulties in principle.

In the entire preceding exposition we have used essentially the tight-binding approximation, the validity of which was ensured by the smallness of $\beta, \beta_1, \ldots$. This calls for two remarks.

1. At $V_n \gg 1$ the tight-binding approximation is violated for Eq. (B), while the quasiclassical approximation is violated at $V_n \gtrsim 1/\beta_n$ for Eq. (A). Therefore, in Eq. (1), on the succeeding levels, $\cos(2\pi\beta_k x)$ is replaced by a more complicated function of x, and modulation of the overlap integrals appears as well. Nonetheless, the estimate (7c), in which V_k must be taken to mean the ratio of the characteristic value of the potential to the characteristic value of the overlap integral, remains in force in this case, too.

Indeed, assume for the sake of argument that $V \gg 1$. The Schrödinger equation on the first rung will have the form (19), where $\varepsilon_s^A(p,\varphi)$ is one of the n_1 solutions of the n_1-th-order secular equations, which, as can be easily shown, has the following structure:

$$\cos p + Q_{n_1}(E) = 0, \tag{21}$$

where Q_{n_1} is a polynomial of degree n_1 in E, with the highest-order coefficient ~ 1. As $V \to \infty$, the bands of the Schrödinger equation degenerates into the levels

$$E_s = V\cos(2\pi s/n_1 + \varphi), \quad s = 1, \ldots, n_1,$$

and the Wannier functions are localized near the corresponding lattice sites. Consequently the roots of the polynomial $Q_{n_1}(E)$ at large V are close to E_s. We then easily obtain from (21) that the bands have a width $\sim 1/V^{n_1-1}$, whereas their displacement with changing φ is of the order of V. Consequently, the ratio of the characteristic value of the potential to the value of the overlap integral (which is none other than V_1) is of the order of V^{n_1}, which agrees with (7c).

The validity of the estimate (7c) is sufficient for the conclusion that localization takes place; to calculate the critical exponent, however, all that is important is the form of the transformation (6) at $V_n \approx 2$. A similar remark must be made for $V_n \ll 1$.

2. At $V_n \approx 2$ the strong-coupling approximation cannot be used near the center of an n-th order band, therefore the theory expounded is valid for not all states. Nonetheless, $V = 2$ is the threshold of localization for the entire band.

It is easiest to verify this using the following heuristic considerations.[5] On going into the momentum representation, Eq. (A) is transformed into itself with the substitution $V/2 \to 2/V$ [see (B)], i.e., the equation with $V > 2$ goes over into the equation with $V < 2$ and vice versa. Localization in coordinate space corresponds to delocalization in momentum space and vice versa, therefore if the threshold value of V exists, it cannot be anything but two. That this reasoning is not rigorous is clear even from the fact that the case of rational β is not singled out in any way. Nonetheless, Eq. (A) has an additional symmetry at $V = 2$; let us show how this symmetry can be used consistently.

First, it is easy to show that there exists a localization threshold for all states; indeed, from the remark one follows exponential localization of all states at $V \gg 1$, from which we obtain, after a Fourier transformation, the delocalization of all the states at $V \ll 1$.

Next, at $V = 2$, Eqs. (A) and (B) coincide, so that the functions $\varepsilon_s^A(p,\varphi)$ and $\varepsilon_s^B(p,\varphi)$ are identical. It follows then from (20) that the function $\varepsilon_s^A(p\varphi)$ is symmetrical with respect to permutation of p and φ, and consequently, in view of (19), that the Hamiltonian is invariant on the next level of the devil's stairs relative to the permutation of $\hat{p}$ and x. Therefore Eqs. (A) and (B) will coincide also on this level. Continuing the induction, we find that the invariance of the Hamiltonian with respect to permutation of $\hat{p}$ and x is preserved on all the succeeding levels of the devil's stairs.

The Hamiltonian $\varepsilon_s^A(\hat{p},\beta_1 x)$, generally speaking, cannot be represented in the form of two terms corresponding to potential and kinetic energy. It is clear, however, that the meaning of the potential V_1 is possessed by the characteristic oscillation amplitude $\varepsilon_s^A(p,\beta_1 x)$ when x is varied, whereas the order of magnitude of the overlap integral J_1 is determined by the amplitude of the oscillations in p. From the established symmetry of $\varepsilon_s^A(p,\varphi)$ it is clear that $V_1 \sim J_1$. Similarly, $V_n \sim J_n$ on an arbitrary n-th level, and in particular as $n \to \infty$. This situation can take place only at the transition point, for otherwise the ratio V_n/J_n should tend as $n \to \infty$ either to zero or to infinity (we assume, of course, that the renormalization-group transformation (6) has qualitatively the same properties as under conditions for the applicability of the tight-binding approximation). Thus, $V = 2$ is the Anderson transition point for all states.

3. CERTAIN GENERALIZATIONS

In view of the x, p symmetry of Eq. (1), indicated in the preceding section, there is no data that for the model described by this equation $V = 2$ is the Anderson transition point at arbitrarily (and not only small) $\beta, \beta_1, \beta_2, \ldots$. The existence of a single threshold for all states is a specific property of this model.

In the general case the incommensurate system is described by a difference equation with a periodic coefficient of more general form than (1), namely, account must be taken of overlap integrals with a larger number of nearest neighbors, as well as their modulation, which, just as the modulation of the levels, can be an arbitrary periodic function. At $\beta \ll 1$, however, this case reduces to the one considered, inasmuch as the tight-binding approximation will be applicable to the first-order bands, and they will be described by Eq. (1). Consequently, localization of each of the first-order bands will take place at a definite value of the parameter V, but these values will be different for the different bands. For the initial band there will therefore exist two critical values V_{c1} and V_{c2} such that at $V > V_{c2}$ all the states are localized, and at $V < V_{c1}$ all are delocalized, whereas in the interval $V_{c1} < V < V_{c2}$ there are mobility energy thresholds.

For noncommensurate systems of general form and at not small $\beta, \beta_1, \beta_2, \ldots$ the situation becomes much more complicated. Now the localization of each of the bands of first order will take place also in a finite interval of values of V; the same holds true also for bands of second etc. order. Therefore, the structure of the mobility threshold will take the

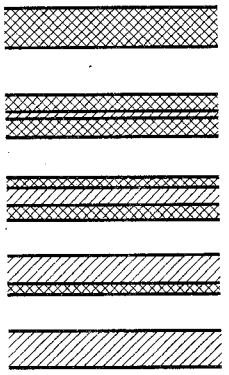

FIG. 4. Structure of mobility threshold in incommensurate systems of general form. Several bands of first order are shown, and the single and double hatchings show localized and delocalized states, respectively. The mobility thresholds in the first-order bands are constructed in the same manner.

form shown in Fig. 4. The mobility threshold in bands of higher order are similarly constructed. Thus, an ever finer mixing of the localized and delocalized states takes place; it stops on the rung where the small β_k first appears.[2)]

In the latter case ($\beta,\beta_1,\ldots,\beta_{k-1}\sim 1$, $\beta_k \ll 1$) it is expedient, when calculating the mobility threshold, to "jump over" all the steps with small β_n, i.e., to represent β in the form

$$\beta = p_k/(q_k+\gamma), \quad \gamma\sim\beta_k/p_k \ll 1$$

(p_k and q_k are integers). Next, solving the secular equation of order k_k, we write down Eqs. (19) with $\varphi = \gamma x$. For these equations it is already possible to use the quasiclassical approximation.

In the same manner, in all the essential cases, the description of incommensurate system reduces to Eq. (1). On the other hand, the only qualitative difference between the general type of system from the model of Sec. 1 is in the dependence of the critical value of V on the energy.

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[1)]The set of numbers of this type has the cardinality of a continuum. By way of example we indicate the irrationalities of the form $\beta = (n^2+1)^{1/2} - n$, $n \gg 1$.

[2)]Or else it continues to infinity if none of the β_k is small. This, however, is not very likely, since the set of irrational numbers with all $\beta_k \sim 1$ (i.e., bounded by n_k) had a zero measure.[9]

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