Density of states near an Anderson transition in four-dimensional space: Lattice model

I. M. Suslov

P. N. Lebedev Physics Institute, Russian Academy of Sciences, 117924 Moscow, Russia
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Asymptotically exact results for the average Green's function and the average density of states of a disordered system, described by the Anderson model with a Gaussian site energy distribution in four-dimensional space, are derived in the limit of weak disorder. The approximation employed consists of taking into account (a) the parquet terms containing the maximum power of the large logarithm and (b) the most rapidly growing (in the limit $N\to\infty$, where N is the order of the perturbation theory) terms with low powers of the logarithms. The latter terms are calculated by the Lipatov method and lead to a nonperturbative contribution to the imaginary part of the Green's function, which accounts for elimination of the "spurious pole".

1. INTRODUCTION

By analogy with the modern theory of critical phenomena[1–3] it is natural to expect that an upper critical space dimension d_{c2} near which an ε-expansion can be constructed exists for the Anderson transition.[4–7] Until recently the question of the value of d_{c2} remained controversial: Although many facts[8,9] pointed to dimension $d_{c2}=4$, they also admitted a different interpretation,[10] which led to the alternative hypotheses $d_{c2}=6$ (Ref. 11), $d_{c2}=8$ (Refs. 12 and 13), and others. A constructive solution to the problem of the upper critical dimension was proposed recently in Ref. 14: It was asserted that $d_{c2}=4$, it was shown how the condition $d>4$ simplifies the problem, and the density of states of the disordered system was calculated for $d>4$ in the entire energy range, including near the Anderson transition. In the present paper the next step in the construction of the ε-expansion is made and the case $d=4$ is examined.

In what follows we shall have in mind the Anderson model on a d-dimensional cubic lattice described by the discrete Schrödinger equation

$$\sum_{x'} J_{x-x'}\Psi_{x'}+V_x\Psi_x=E\Psi_x \tag{1}$$

with a Gaussian distribution of the site energy V_x

$$P\{V\}\sim\exp\left\{-\sum_x V_x^2/2W^2\right\}. \tag{2}$$

The disorder amplitude W and the energies E considered are assumed to be small compared to the width J of the band; the energy origin and the overlap integrals $J_{x-x'}$ are chosen so that the spectrum of the ideal lattice

$$\varepsilon(k)=\sum_x J_x e^{-ikx} \tag{3}$$

has the form $k^2/2m$ for small k. In the continuous limit, when the lattice constant satisfies $a_0\to 0$ with $2m=\text{const}$ and $a_0^d W^2=\text{const}$, the problem of calculating the average Green's function of Eq. (1) is mathematically equivalent to the problem of a second-order phase transition in a system with an n-component order parameter $\varphi=(\varphi_1,\varphi_2,...,\varphi_n)$ in the limit $n\to 0$: This can be shown by comparing the diagrammatic expansions (Ref. 2, p. 225) or by switching to a functional integral by means of the replica method.[15] Then the coefficients in the Ginzburg–Landau Hamiltonian

$$H\{\varphi\}=\int d^dx\left(\frac{1}{2}c\left|\nabla\varphi\right|^2+\frac{1}{2}\kappa_0^2\left|\varphi\right|^2+\frac{1}{4}g\left|\varphi\right|^4\right) \tag{4}$$

are related to the parameters of the disordered system by the relations

$$c=1/2m,\quad \kappa_0^2=-E,\quad g=-a_0^dW^2/2. \tag{5}$$

As usual, the coefficient of $|\nabla\varphi|^2$ is positive and the coefficient of $|\varphi|^2$ changes sign near a phase transition (in the case of weak disorder the Anderson transition lies at low values of E), but the coefficient of $|\varphi|^4$ has the "wrong" sign. Consequently, the results of the theory of phase transitions cannot be transferred directly to the physics of disordered systems. In quantum field theory the model corresponding to (4) is the well-known φ^4 model which describes a relativistic Bose gas with a point interaction;[16,17] here negative values of the interaction constant g (attraction) correspond to an unstable field theory.[1)]

The spatial dimension $d=4$ is distinguished for the Hamiltonian (4): For $d=4$ there exists a dominant sequence of diagrams which contain the maximum power of a large logarithm—the so-called parquet.[18–20] Summation of the parquet terms leads to the following relation between the renormalized constant g_R, describing the interaction at large distances, and the bare value g (Refs. 3, 19, 21):

$$g_R=\frac{g}{1+K_4(n+8)g\,\ln(\Lambda/\kappa)}, \tag{6}$$

where $K_4=(8\pi^2)^{-1}$ is the area of the four-dimensional unit sphere divided by $(2\pi)^4$; Λ is the large-momentum cutoff parameter; and κ is the renormalized value of κ_0, which vanishes at the transition point. For $g>0$ the expression (6) describes the typical "zero-charge" situation:[16,22,23] As $\kappa\to 0$, i.e., as the phase transition is approached, the effective interaction approaches zero. In the transition from $d=4$ to

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$d=4-\varepsilon$ the constant g_R acquires in the limit $\kappa\to 0$ a finite but small value, and this is actually why Wilson's ε-expansion is successful.[1,2]

The use of the parquet approximation in the theory of disordered systems leads to an expression of the form (6) with $g<0$;[24,25] for small κ (i.e., small values of the renormalized energy E) the expression (6) has a "spurious" pole that cannot be removed within the parquet approximation. The formal application of parquet results of the type (6) leads to divergences in physical quantities. Thus a paradoxical situation arises: The use of the same approximation in two mathematically equivalent problems leads in one case (theory of phase transitions) to essentially a complete solution of the problem, and in the other case (theory of disordered systems) to manifestly unphysical results. Sadovskii[24,25] believes that the spurious pole problem is the main obstacle in the construction of a systematic theory of the Anderson transition. In the present paper a solution of this problem is given for a definite class of lattice models.

The source of the difficulties is actually not so much the incorrect sign of the interaction constant g as the irremovable complexity of the effective Hamiltonian (4). An infinitesimal imaginary part $+i\delta$ or $-i\delta$ must be added to the energy E even to specify the choice of the Green's function (retarded or advanced), and the choice of sign of the imaginary part must be taken into consideration in the analytic continuation in the coupling constant or other method for removing the divergence of the functional integrals (see footnote 1). The instability of the field theory for $g<0$ results in the fact that the infinitesimal "seed" $\pm i\delta$ leads to the appearance of a, generally speaking, finite imaginary part in all calculated quantities. The problem of calculating the imaginary part did not arise in the theory of phase transitions, but it is here that the parquet approximation becomes completely unsatisfactory. The main effect is that the transition "temperature" acquires an imaginary part: When this circumstance is taken into account the mathematical equivalence of the two problems is restored—the physical quantities of the disordered system which are determined by the average Green's function are described by the formulas of the theory of phase transitions with complex T_c; this latter circumstance accounts for elimination of the "spurious pole".

Analysis of the case $d>4$ showed[14] that to resolve the difficulties described above the factorial divergence of the perturbation-theory series must be taken into account correctly;[26] it is this divergence that determines, in particular, the appearance of the fluctuation-induced tail of the density of states.[27,28] The following scheme was employed in the preceding work[14] to investigate the higher orders of perturbation theory: a) the functional form of the coefficients in the expansion of the self-energy Σ, containing a small number of phenomenological parameters, was established by statistical analysis proceeding from the diagram technique; b) the nonperturbative contribution to the damping Γ was calculated, which made it possible to calculate the density of states $\nu(E)$ at all energies; and c) the phenomenological parameters introduced above were determined by comparing the asymptotic behavior of $\nu(E)$ in the limit $E\to-\infty$ to the results obtained by the instanton method.[24,25,28,29]

In the present paper the same idea is implemented by a more direct method: a) The asymptotic behavior of the coefficients in the expansion of the Green's function is calculated by the Lipatov instanton method;[26] b) the asymptotic behavior of the coefficients of the self-energy are found from it; c) the form of the self-energy $\Sigma(p,E)$ for arbitrary p and E is determined by comparing the instanton results, which are valid for $E<0$, to the structure of the perturbation-theory series following from the diagrammatic technique; and d) $\nu(E)$ is calculated. In this approach a statistical analysis, some aspects of which are conditional, of the diagrams is no longer required;[14] however, substantial use is made of the fact that, as demonstrated in Ref. 14, the instanton results can in principle be obtained by the diagrammatic technique.

In the discussion below units with $a_0=1$ and $2m=1$ are employed; the standard units are used only for estimates.

2. STRUCTURE OF THE PERTURBATION SERIES FOR $d=4$

For convenience in applying the published results the exposition will be given in terms of the theory of phase transitions, and the transition to disordered systems will be made at a later stage.

We define the M-point Green's function as

$$G_M(\alpha_1,x_1,\ldots,\alpha_M,x_M)=Z_0^{-1}Z_M(\alpha_1,x_1,\ldots,\alpha_M,x_M), \tag{7}$$

where Z_M is given by the functional integral

$$Z_M(\alpha_1,x_1,\ldots,\alpha_M,x_M)=\int D\varphi\varphi_{\alpha_1}(x_1)\ldots\varphi_{\alpha_M}(x_M)e^{-H\{\varphi\}} \tag{8}$$

and the Hamiltonian $H\{\varphi\}$ in the continuous approximation is determined by the expression (4). The standard diagram technique[2] is obtained by expanding the Green's function (7) in powers of g. The calculation of the two-point Green's function $G(\alpha_1x_1,\alpha_2x_2)=\delta_{\alpha_1\alpha_2}G(x_1-x_2)$ reduces, in the standard fashion, to calculating the self-energy Σ:

$$G^{-1}(k)=\varepsilon(k)+\kappa_0^2-\tilde{\Sigma}(k,\kappa_0^2)=\varepsilon(k)+\kappa^2-\Sigma(k,\kappa^2)+\Sigma(0,\kappa^2), \tag{9}$$

where $G(k)$ is the Fourier transform of $G(x_1-x_2)$, and the quantity κ^2 is

$$\kappa^2=\kappa_0^2-\tilde{\Sigma}(0,\kappa_0^2)=\kappa_0^2-\Sigma(0,\kappa^2). \tag{10}$$

The transition point is determined by the condition $G^{-1}(0)=0$ or $\kappa_0=\kappa_c$, where

$$\kappa_c^2=\Sigma(0,0). \tag{11}$$

Counting the powers of the momentum in the Nth order diagram for Σ gives $r=2+(d-4)N$, and for $d=4$ all diagrams diverge at the upper limit as Λ^2. Subtracting from each diagram its value at $k=0$, $\kappa=0$ decreases the exponent r by 2 and makes the divergence logarithmic. The quadratic divergences are removed from the inner blocks by switching from the bare Green's functions $G_B(k)=(\varepsilon(k)+\kappa_0^2)^{-1}$ to the modified functions

$$G_0(k)=(\varepsilon(k)+\kappa^2)^{-1} \tag{12}$$

(when the expansion is made in terms of the functions $G_0(k)$ the diagrams have the same form as in the case of the expansion in terms of $G_B(k)$ but from each irreducible self-energy part its value at zero momentum is subtracted). As a result, all quadratic divergences are incorporated into the term $\Sigma(0,0)$, and the difference $\Sigma(k,\kappa^2)-\Sigma(0,0)$ contains only logarithmic divergences; classifying the contributions of the diagrams according to powers of the logarithms we obtain for $k=0$

$$\Sigma(0,\kappa^2)=\Sigma(0,0)+\kappa^2\sum_{N=1}^{\infty} g^N\sum_{K=0}^{N} A_N^K\left(\ln\frac{\Lambda}{\kappa}\right)^K, \qquad (13)$$

$$\Sigma(0,0)=\Lambda^2\sum_{N=1}^{\infty} B_N g^N. \qquad (14)$$

The maximum power of the logarithm is determined by the parquet diagrams and equals the order of the diagram. In Eq. (13) terms of the type κ^4/Λ^2, κ^6/Λ^4, and so on, which vanish in the limit $\Lambda\to\infty$, are dropped.

The parquet approximation corresponds to retaining in Eq. (13) only the coefficients A_N^N. The corresponding result can be easily derived from the work of Ginzburg.[21] From the Ward identity we have

$$\frac{\partial G^{-1}(0,\kappa^2)}{\partial\kappa_0^2}=\frac{\partial\kappa^2}{\partial\kappa_0^2}=\left(1+K_4(n+8)g\ln\frac{\Lambda}{\kappa}\right)^{-(n+2)/(n+8)} \qquad (15)$$

(see Eqs. (32), (30), and (17) of Ref. 21 in the limit $\varepsilon\to 0$). Integrating Eq. (15) in the leading logarithmic approximation and using Eq. (10) we obtain

$$\Sigma(0,\kappa^2)=\Sigma(0,0)+\kappa^2\times\left[\left(1+K_4(n+8)g\ln\frac{\Lambda}{\kappa}\right)^{(n+2)/(n+8)}-1\right]. \qquad (16)$$

Expanding in a series in powers of g and comparing to Eq. (13) we have

$$A_N^N=[-K_4(n+8)]^N\times\frac{\Gamma(N-\beta)}{\Gamma(N+1)\Gamma(-\beta)}\xrightarrow{N\to\infty}\frac{1}{\Gamma(-\beta)}\times[-K_4(n+8)]^N N^{-\beta-1}, \qquad (17)$$

where $\beta=(n+2)/(n+8)$ and $\Gamma(x)$ is the gamma function.

The analysis of the case $d>4$ performed in Ref. 14 showed the qualitative importance of taking into account the factorial divergence of the perturbation series. This divergence is directly related to the existence of a fluctuation-induced tail of the density of states. Indeed, for a Gaussian site-energy distribution (2) arbitrarily deep fluctuations of the potential and therefore arbitrarily deep energy levels with arbitrarily small W^2 occur with finite probability, i.e., the density of states $\nu(E)$ is different from zero for all E and W^2. Hence

$$G(E+i\delta)-G(E-i\delta)=\text{const}\cdot\nu(E)\neq 0 \quad \text{for all } E,\ g<0, \qquad (18)$$

i.e., the exact Green's function $G(E)$ has a cut for negative values of E that the unperturbed Green's function $G_0(E)$ did not have. According to a well-known theorem from analysis the sum of a series consisting of continuous functions is continuous if the series converges uniformly; uniform convergence obtains if the functional series is majorized by a converging numerical series.[30] If the coefficients in the expansion of $G(E)$ in powers of g grow more slowly than a^N for some finite a, then for small $|g|$ the series is majorized by a converging geometric progression and converges uniformly. Then the impossibility of Eq. (18) follows from the continuity, following from the continuity of $G_0(E)$, of the terms of the series. For this reason, the expansion coefficients grow more rapidly than a^N with arbitrarily large a: The factorial function is one of the simplest functions with this property.

It is clear from Eq. (17) that the coefficients of the parquet expansion do not exhibit factorial growth. This reveals the reason that the parquet expansion is unsatisfactory. Terms with low powers of the logarithms grow more rapidly and dominate for large values of N.

In principle a second, third, and so on logarithmic approximation can be made by retaining in Eq. (13) the coefficients A_N^{N-1}, A_N^{N-2}, and so on. These coefficients are determined implicitly when the higher order terms of the ε-expansion are calculated by Wilson's method, which is based on the existence of an exact renormalization group for $d=4-\varepsilon$, $g>0$ (Ref. 2, chapter 9). For finite K and $N\to\infty$

$$A_N^{N-K}=\text{const}\cdot[-K_4(n+8)]^N N^{-\beta-1}(N\ln N)^K. \qquad (19)$$

We can see that factorial growth does not arise in any finite logarithmic approximation; it can be seen for $K\sim N$, when Eq. (19) is no longer applicable.

Information about the most rapidly growing coefficients can be obtained by the Lipatov method (Sec. 4). These coefficients correspond to the terms with low powers of the logarithms and depend on the character of the cutoff at large momenta (for example, changing Λ by a factor of 2 changes all A_N^K with $K\neq N$), and therefore on the choice of model. As a guide, we shall make some simple estimates.

3. ESTIMATES BY THE OPTIMAL-FLUCTUATION METHOD AND CLASSIFICATION OF MODELS

It is clear from Sec. 2 and Ref. 14 that the situation in the higher orders of the perturbation theory is associated with the appearance of a fluctuation-induced tail, which in the field-theoretic formulation is determined by the classical solutions (instantons) in the density of states.[24,25,28,29] The properties of instantons can be understood qualitatively by the optimal-fluctuation method of Lifshitz.[31,32]

On account of Eq. (2) the probability of the appearance of a fluctuation-induced potential well with depth V and radius R is of order[2)]

$$P(V,R)\sim\exp\{-V^2R^d/W^2a_0^d\}. \qquad (20)$$

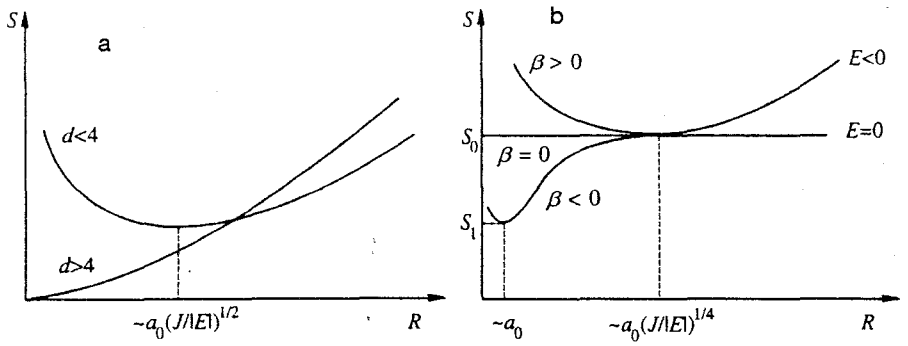

FIG. 1. $S(E,R)$ as a function of R with $E=$const: (a) $d\neq 4$ and (b) $d=4$.

When a level $E=-|E|$ is present in the well the parameters V and R are related by

$$E=-V+(1/mR^2)\simeq -V+J(a_0/R)^2, \tag{21}$$

which makes it possible to eliminate V from Eq. (20):

$$P(E,R)\propto \exp\left[-\left(\frac{R}{a_0}\right)^d\left(\frac{|E|+J(a_0/R)^2}{W}\right)^2\right]$$

$$\equiv \exp\{-S(E,R)\}. \tag{22}$$

The level-formation probability $P(E)$, which determines the density of states $\nu(E)$, is obtained by integrating Eq. (22) over R. In the saddle-point approximation the contribution to the integral is determined by the point $R=R_c$, where the quantity $S(E,R)$ is minimum. The quantity R_c corresponds to the instanton radius in the field-theoretic formulation: It is clear from Fig. 1a that for $d<4$ the instanton radius $R_c\propto|E|^{-1/2}$ and diverges as $|E|\to 0$, while for $d>4$ it corresponds to the minimum possible R, i.e., $R_c\sim a_0$.

For $d=4$ (Fig. 1b) we have $S(E,R)=\text{const}=S_0$ for $E=0$, and the situation is close to degeneracy. For large R the degeneracy is removed due to the finiteness of E, $S(E,R)\propto E^2R^4$; for small R the behavior of the spectrum $\varepsilon(k)$ for large k becomes important: If in the expansion of $\varepsilon(k)$ in powers of k terms $\sim k^4$ are included together with terms $\sim k^2$, then instead of Eq. (21) we obtain

$$E=-V+J(a_0/R)^2+\beta J(a_0/R)^4. \tag{23}$$

For $\beta>0$ (repulsive "core") the quantity $S(E,R)$ is larger than S_0, giving rise to a minimum at $R\sim a_0(J/|E|)^{1/4}$, while for $\beta<0$ (attractive "core") $S(E,R)$ is less than S_0 and a minimum obtains at $R\sim a_0$, where higher order terms in the expansion of $\varepsilon(k)$ in powers of k become important (Fig. 1b). Thus the transition from the highest dimensions of the space to the lowest dimensions is "continued" at $d=4$ in the parameters of the model: For $\beta<0$ the instanton is localized on an atomic scale, similarly to the case $d>4$, and for $\beta>0$ the instanton radius diverges as $E\to 0$, which is characteristic for the lowest dimensions. Correspondingly, the asymptotic behavior of the density of states for large negative E is found to be different:

$$\nu(E)\sim \begin{cases} \exp\left[-\dfrac{J^2}{W^2}\left(1+\dfrac{|E|}{J}\right)\right], & \beta<0 \\ \exp\left[-\dfrac{J^2}{W^2}\left(1+\dfrac{|E|^{1/2}}{J^{1/2}}\right)\right], & \beta>0. \end{cases} \tag{24}$$

The sharp difference between the attractive and repulsive cores disappears for small values of E, where Anderson's transition lies. The point is that for $\beta<0$ the contribution of the instanton minimum with $S=S_1$ competes with that of the higher-lying plateau with $S=S_0$, whose width increases without bound as $E\to 0$. Integrating $P(E,R)$ over R, taking into account both contributions, gives a result of the form

$$\nu(E)\sim \exp(-S_1)+\left(\frac{J}{|E|}\right)^{1/2}\exp(-S_0) \tag{25}$$

and as S_1 and S_0 approach one another the second term (the plateau contribution) inevitably dominates, so that the transition to the case $\beta>0$, accompanied by vanishing of the first term in Eq. (25), has no consequences of importance.

Since the energy E always appears in the combination[3)] $E+i\Gamma$ with the damping Γ, near the Anderson transition we must set $E\sim\Gamma$ in Eq. (25). Then two limiting cases can be distinguished.

1) Strongly attractive core. If S_1 is sufficiently smaller than S_0, the first term dominates in Eq. (25). In this case Γ is determined by the corresponding exponential $\exp\{-S_1\}$, which ensures that the second term is small. The contribution to $\nu(E)$ is determined by instantons localized on an atomic scale and the discreteness of the lattice is of fundamental importance.

2) Weakly attractive (S_1 close to S_0), repulsive, or neutral ($\beta=0$) core. The density of states is determined by the second term in Eq. (25)—by the contribution of the plateau. Since large-radius instantons are important, the atomic structure of the lattice is of no importance and the continuous model ($a_0\to 0$) can be used for the analysis.

In the present paper we confine our attention to the first case, in which, since the model is discrete, ultraviolet divergences do not arise. The second case is more complicated: Here, just as in quantum electrodynamics, three quantities must be renormalized in passing to the continuous limit—"charge," "mass," and the Green's function. This renormalization must be performed consistently in higher orders of perturbation theory.

To prove the existence of models with small S_1 we shall estimate the instanton minimum by a variational method for a nonoptimal fluctuation localized at a single site. Setting in Eq. (2) $V_x=V\delta_{x0}$ and eliminating V by means of the equation

$$1+VG_{00}(E)=0 \tag{26}$$

determining the level in this potential[31] ($G_{xx'}(E)$ is the Green's function of an ideal lattice), we obtain in the limit $E\to 0$

$$S_1<[2W^2G_{00}^2(0)]^{-1}=\frac{1}{2W^2}\left[a_0^d\int\frac{d^dk}{(2\pi)^d}\frac{1}{\varepsilon(k)}\right]^{-2}$$
$$<\frac{J^2}{2W^2}$$

if the spectrum $\varepsilon(k)$ lies in the interval from 0 to J. The value of S_0 for the plateau is determined by the effective mass m for small k. This effective mass is not explicitly related to the width J of the band. The ratio S_1/S_0 can be made arbitrarily small by decreasing J with m=const.

4. ASYMPTOTIC FORM OF THE EXPANSION COEFFICIENTS FOR LARGE N

4.1. Application of the Lipatov method

The Lipatov method[26] is based on the fact that the expansion coefficients of the function $F(g)$

$$F(g)=\sum_{N=0}^{\infty}F_Ng^N \tag{27}$$

can be determined from the formula

$$F_N=\int_C\frac{dg}{2\pi i}g^{-N-1}F(g), \tag{28}$$

where the contour C encircles the point $g=0$ in the complex plane, and for large N the integral in Eq. (28) can be calculated by the saddle-point method. Substituting for $F(g)$ the quantity Z_M from Eq. (8) gives

$$[Z_M]_{N-1}=\int\frac{dg}{2\pi i}\int D\varphi\varphi_{\alpha_1}(x_1)\dots\varphi_{\alpha_M}(x_M)\times$$
$$\exp(-H\{\varphi\}-N\ln g), \tag{29}$$

where $H\{\varphi\}$ is the lattice-model Hamiltonian

$$H\{\varphi\}=\sum_x\left[\frac{1}{2}\sum_\alpha\varphi_\alpha(x)\varepsilon(\hat p)\varphi_\alpha(x)+\frac{1}{2}\kappa^2\sum_\alpha\varphi_\alpha(x)^2\right.$$
$$\left.+\frac{1}{4}g\left(\sum_\alpha\varphi_\alpha(x)^2\right)^2\right]. \tag{30}$$

The action of the operator $\varepsilon(\hat p)$, where $\hat p=-i\nabla$, is determined by using the Fourier expansion (3) and taking into consideration the fact that the operator $\exp(i\hat{\mathbf{p}}\mathbf{a})$ describes a displacement by the vector $\mathbf{a}$. The idea of the Lipatov method is to use in Eq. (29) the method of steepest descent in g and φ simultaneously. The conditions for the applicability of the method of steepest descent are satisfied for large N and $\kappa^2>0$, irrespective of whether or not this method is applicable to the initial integral (8). The saddle point is determined by the conditions

$$\begin{pmatrix}\partial/\partial g\\ \delta/\delta\varphi\end{pmatrix}\left\{-N\ln g-\sum_x\left[\frac{1}{2}\sum_\alpha\varphi_\alpha\varepsilon(\hat p)\varphi_\alpha+\frac{1}{2}\kappa^2\sum_\alpha\varphi_\alpha^2\right.\right.$$
$$\left.\left.+\frac{1}{4}g\left(\sum_\alpha\varphi_\alpha^2\right)^2\right]\right\}=0. \tag{31}$$

If the classical solution (instanton) is sought in the form

$$\varphi_\alpha^c(x)=\varphi_c(x)u_\alpha, \tag{32}$$

where u_α are the components of the unit vector $\mathbf{u}$, then Eq. (31) gives

$$Ng_c^{-1}=-\frac{1}{4}\sum_x\varphi_c(x)^4, \tag{33}$$

$$\varepsilon(\hat p)\varphi_c(x)+\kappa^2\varphi_c(x)+g_c\varphi_c(x)^3=0, \tag{34}$$

whence it is clear that $g_c<0$, and to guarantee that the sum in Eq. (3) converges the function $\varphi_c(x)$ is sought in a class of functions that decay at infinity. Since a strongly attractive core is assumed, we assume that the instanton is localized on an atomic scale. Considering small deviations from the saddle point

$$g=g_c+\delta g,\quad \varphi_\alpha(x)=\varphi_c(x)u_\alpha+\delta\varphi_\alpha(x) \tag{35}$$

and separating $\delta\varphi_\alpha(x)$ into longitudinal and transverse parts

$$\delta\varphi_\alpha(x)=\delta\varphi_L(x)u_\alpha+\delta\varphi_\alpha^T(x),\quad \delta\varphi^T\perp\mathbf{u} \tag{36}$$

we obtain for the argument of the exponential in Eq. (29)

$$-N\ln g-H\{\varphi\}=-N\ln g_c-H\{g_c,\varphi_c\}+\tfrac{1}{2}N(\delta g/g_c)^2$$
$$-\delta g\sum_x\varphi_c(x)^3\delta\varphi_L(x)$$
$$-\frac{1}{2}\sum_x\delta\varphi_L(x)\hat M_L\delta\varphi_L(x)$$
$$-\frac{1}{2}\sum_x\sum_\alpha\delta\varphi_\alpha^T(x)\hat M_T\delta\varphi_\alpha^T(x), \tag{37}$$

where

$$\hat M_L=\varepsilon(\hat p)+\kappa^2+3g_c\varphi_c(x)^2,\quad \hat M_T=\varepsilon(\hat p)+\kappa^2+g_c\varphi_c(x)^2. \tag{38}$$

Introducing the eigenvalues and the normalized eigenfunctions

$$\hat M_Le_s^L(x)=\lambda_s^Le_s^L(x),\quad \hat M_Te_s^T(x)=\lambda_s^Te_s^T(x) \tag{39}$$

and expanding $\delta\varphi_L$ and $\delta\varphi_\alpha^T$ in terms of these eigenfunctions as

$$\delta\varphi_L(x)=\sum_s C_s^Le_s^L(x),\quad \delta\varphi_\alpha^T(x)=\sum_s C_s^{T,\alpha}e_s^T(x) \tag{40}$$

we obtain Eq. (37) in the form

$$-N\ln g-H\{\varphi\}=-N\ln g_c-H\{g_c,\varphi_c\}$$
$$+\frac{1}{2}N(\delta g/g_c)^2-(\delta g/g_c)\sum_s C_s^L\omega_s$$

$$-\frac{1}{2}\sum_g \lambda_s^L (C_s^L)^2$$

$$-\frac{1}{2}\sum_s \sum_\alpha \lambda_s^T (C_s^{T,\alpha})^2, \qquad (41)$$

where

$$\omega_s = g_c \sum_x \varphi_c(x)^3 e_s^L(x). \qquad (42)$$

The integration over $\varphi(x)$ can be replaced by integration over C_s^L and $C_s^{T,\alpha}$, the determinant of the transformation being equal to 1 since e_s^L and e_s^T are orthonormal:

$$\int D\varphi = \int DC^L \int DC^T. \qquad (43)$$

4.2. Treatment of zero modes

In the continuous model the operator $\hat{M}_L$ has zero modes associated with the possibility of continuous translations of the instanton solution.[25] In the discrete model these become gap modes and they need not be specially distinguished; however, in order to obtain a translationally invariant result, it is convenient to carry out a procedure similar to the one employed in making such a distinction. We introduce in the integrand of Eq. (29) the partition of unity $\Sigma_y \delta_{y-y_0}$, identifying y_0 with the site near which the instanton is localized, and we perform the transformation

$$x - y_0 + y \to x, \quad \varphi(x + y_0 - y) \to \varphi(x), \qquad (44)$$

which displaces the instanton to the site y (after which $\delta_{y-y_0} = 1$). Since $D\varphi$ and $H\{\varphi\}$ are invariant under the transformation (44), the integral in Eq. (29) assumes the form

$$[Z_M]_{N-1} = \int \frac{dg}{2\pi i} \int D\varphi \sum_y \varphi_{\alpha_1}(x_1 - y) \ldots \varphi_{\alpha_M}(x_M - y) \times \exp(-H\{\varphi\} - N \ln g). \qquad (45)$$

To separate the zeroth mode of the operator $\hat{M}_T$ (Ref. 25)

$$\lambda_0^T = 0, \quad e_0^T = \left(\sum_x \varphi_c(x)^2\right)^{-1/2} \varphi_c(x) \qquad (46)$$

associated with the possibility of rotating the vector $\mathbf{u}$ in Eq. (32), we introduce in the integrand in Eq. (47) the partition of unity

$$1 = \int d^n u' \prod_{\alpha=1}^{n} \delta(u'_\alpha - v_\alpha) = \int d^n u\, \delta(|\mathbf{u}| - 1) \times \left(\sum_x \varphi_c(x)^2\right)^{(n-1)/2} \prod_{\alpha=1}^{n-1} \delta(C_0^{T,\alpha}), \qquad (47)$$

where the unit vector $\mathbf{v}$ is chosen so that

$$\mathbf{v} \| \sum_x |\varphi(x)| \varphi(x). \qquad (48)$$

To obtain the second relation in Eq. (47) we must identify the vector $\mathbf{u}'$ with the vector $\mathbf{u}$ in Eq. (32) (which is possible because of the arbitrariness of the latter), express v in terms of $\varphi_c(x)$ and $\delta\varphi(x)$, and use Eq. (40) and the fact that $e_s^T(x)$ with $s \neq 0$ is orthogonal to $e_0^T(x)$. Substituting Eq. (47) into the integrand in Eq. (45), using the zeroth-order approximation (32) of $\varphi_\alpha(x)$ in the pre-exponential factor, using Eqs. (41) and (43), and removing the δ-function in Eq. (47) by integrating over $C_0^{T,\alpha}$, we obtain

$$[Z_M]_{N-1} = \sum_y \varphi_c(x_1 - y) \ldots \varphi_c(x_M - y) \times \int d^n u\, \delta(|u| - 1) u_{\alpha_1} \ldots u_{\alpha_M} \times \left(\sum_x \varphi_c(x)^2\right)^{n-1/2} \exp(-H\{g_c, \varphi_c\} - N \ln g_c) \frac{g_c}{2\pi} \int_{-\infty}^{\infty} dt \int DC^L \int D'C^T \times \exp\left\{-\frac{1}{2}\left[Nt^2 - 2it \sum_s C_s^L \omega_s + \sum_s \lambda_s^L (C_s^L)^2 + \sum_s {}' \sum_\alpha \lambda_s^T (C_s^{T,\alpha})^2\right]\right\}, \qquad (49)$$

where we have made the substitution $\delta g = i|g_c|t$, since the contour C passes through the point g_c in the vertical direction; the prime indicates that the contribution of the zeroth mode (46) is excluded from the sums and products.

4.3. Calculation of the functional integral

The quadratic form in the exponential (49) is nonhermitian and the law of inertia does not hold for it.[33] To reduce it to a diagonal form with positive coefficients, which guarantees that the integral (49) converges, a special transformation of variables must be used. Since the lowest eigenvalue λ_0^L of the operator $\hat{M}_L$ is negative,[25] this transformation can be taken in the form[4)]

$$t^- = t - iC_0^L \omega_0 \left[N + \sum_{s \neq 0} \frac{\omega_s^2}{\lambda_s^L}\right]^{-1},$$

$$\tilde{C}_s^L = C_s^L - it \frac{\omega_s}{\lambda_s^L} \quad (s \neq 0). \qquad (50)$$

To calculate the determinant it is necessary to know the value of the sum

$$\sum_s (\omega_s^2/\lambda_s^L) = \frac{1}{2} g_c \sum_x \varphi_c(x)^4 = -2N \qquad (51)$$

calculated using the completeness of the basis $e_s^L(x)$ and the relation

$$2g_c \sum_x \varphi_c(x)^3 e_s^L(x) = \lambda_s^L \sum_x \varphi_c(x) e_s^L(x), \qquad (52)$$

which is obtained by multiplying Eq. (34) by $e_s^L(x)$ and the first equation in Eq. (39) by $\varphi_c(x)$, and summing over x the equations obtained and taking the difference. Using Eqs. (50) and (51) the result of integration on t, C^L, and C^T in Eq. (49) is equal to:

$$\frac{g_c}{2\pi}\left[\frac{2\pi}{N}\left(\prod_s \frac{2\pi}{|\lambda_s^L|}\right)\left(\prod_s{}' \frac{2\pi}{\lambda_s^T}\right)^{n-1}\right]^{1/2}. \tag{53}$$

To separate the N-dependence we make the substitution

$$\varphi_c(x)=(-g_c)^{-1/2}\psi_c(x), \tag{54}$$

which eliminates g_c from Eqs. (34) and (38), and we introduce the notation

$$I_p=\sum_x \psi_c(x)^p. \tag{55}$$

Then

$$g_c=-I_4/4N, \quad H\{\varphi_c,g_c\}=-\frac{1}{4}g_c\sum_x \varphi_c(x)^4=N \tag{56}$$

and Eq. (49) assumes the form

$$\begin{aligned}[Z_M]_N=&\frac{(-1)^N}{2\pi}\Gamma\left(N+\frac{M+n-1}{2}\right)\\ &\times\left(\frac{4}{I_4}\right)^{N+(M+n-1)/2}\left(\prod_s \frac{2\pi}{|\lambda_s^L|}\right)^{1/2}\\ &\times\left(\prod_s{}' \frac{2\pi}{\lambda_s^T}\right)^{(n-1)/2} I_2^{(n-1)/2}\\ &\times\sum_y \psi_c(x_1-y)\ldots\psi_c(x_M-y)\\ &\times\int d^n u\,\delta(|u|-1)u_{\alpha_1}\ldots u_{\alpha_M}.\end{aligned} \tag{57}$$

4.4. Expansion for the Green function

It is obvious from Eq. (57) that the expansion coefficients of Z_M increase as $N\to\infty$ according to a factorial law. For factorial series there exists a simple algebra (see Appendix) that makes it possible to multiply one series by another, construct integer and fractional powers of a series, and so on. Proceeding from the definition (7) and using the fact that, according to Eq. (57), the sequence $[Z_M]_N$ with $M>0$ increases more rapidly than $[Z_0]_N$ we find that the coefficients in the expansion of the Green's function are the same as the coefficients of the quantity

$$Z_0^{-1}(g=0)Z_M(g)=\left(\prod_s \frac{2\pi}{\lambda_s^0}\right)^{-n/2} Z_M(g), \tag{58}$$

where λ_s^0 are the eigenvalues of the operator $\hat{M}_0=\varepsilon(\hat{p})+\kappa^2$. Expressing the determinants as

$$D_L=\prod_s \lambda_s^L, \quad D_T=\prod_s \lambda_s^T, \quad D_0=\prod_s \lambda_s^0 \tag{59}$$

we obtain the following expressions for the coefficients in the expansion of the Green's function:

$$\begin{aligned}&(G_M(x_1,\alpha_1,\ldots,x_M,\alpha_M))_N\\ &=\frac{(-1)^N}{(2\pi)^{(n+1)/2}}\Gamma\left(N+\frac{M+n-1}{2}\right)\left(\frac{4}{I_4}\right)^{N+(M+n-1)/2}\\ &\times\left[\frac{D_0}{|D_L|}\left(I_2\frac{D_0}{D_T'}\right)^{n-1}\right]^{1/2}\sum_y \psi_c(x_1-y)\ldots\psi_c(x_M-y)\\ &\times\int d^n u\,\delta(|u|-1)u_{\alpha_1}\ldots u_{\alpha_M}.\end{aligned} \tag{60}$$

4.5. Brezin–Parisi transformation of the determinants

The spectrum of the operators $\hat{M}_L$ and $\hat{M}_T$ contains a continuous component, and to calculate the determinants D_L and D_T it is necessary to perform quantization in a large but finite volume and then to pass to the thermodynamic limit. As a rule, however, λ_s^L and λ_s^T cannot be calculated analytically, and in this situation numerical methods are ineffective. Brezin and Parisi proposed a method for overcoming these difficulties.[27] This method also reveals divergences in the determinants, and the elimination of these divergences reveals a connection to the general problems of renormalizability. Introducing the notation

$$D(z)=\det\hat{R}(z), \quad \hat{R}(z)=1-\frac{3z\psi_c(x)^2}{\varepsilon(\hat{p})+\kappa^2} \tag{61}$$

and employing the fact that the determinant of a product equals the product of the determinants, we obtain for the combinations appearing in Eq. (60)

$$\begin{aligned}&\frac{D_L}{D_0}=\frac{\det\{\varepsilon(\hat{p})+\kappa^2-3\psi_c(x)^2\}}{\det\{\varepsilon(\hat{p})+\kappa^2\}}=D(1),\\ &\frac{D_T}{D_0}=D(1/3).\end{aligned} \tag{62}$$

The spectrum of the operator $\hat{R}(z)$ is purely discrete: Its lowest eigenvalues can be found numerically, and simple asymptotic expressions exist for the higher-order eigenvalues (see below). It is easily shown that

$$D(z)=\prod_s\left(1-\frac{z}{\mu_s}\right), \tag{63}$$

where μ_s are the eigenvalues of the equation

$$(\varepsilon(\hat{p})+\kappa^2-3\mu_s\psi_c(x)^2)e_s(x)=0, \tag{64}$$

i.e., μ_s is the value of μ in the potential $-3\mu\psi_c(x)^2$ for Schrödinger's equation with the spectrum $\varepsilon(p)$, in which the sth energy level equals $-\kappa^2$; because of Eq. (63) D_L and D_T can be expressed in terms of a single sequence μ_s. To eliminate the zeroth eigenvalue from D_T it is necessary to determine the manner in which the corresponding eigenvalue of $\hat{R}(z)$ approaches zero as $z\to 1/3$. This can be done by means of perturbation theory:

$$\frac{D_T'}{D_0}=\frac{I_2}{I_4}\tilde{D}(1/3), \quad \tilde{D}(1/3)=\lim_{z\to 1/3}\frac{D(z)}{1-3z}. \tag{65}$$

When Eqs. (62) and (65) are substituted into the expression (60), the quantity I_2, which diverges at $d=4$, $\kappa^2=0$, cancels out (see below).

4.6. Divergences of the determinants and elimination of these divergences

Asymptotic representations of μ_s for large s can be obtained by calculating the number of electrons with the spectrum $\varepsilon(p)$ which have energy less than $-\kappa^2$ in a quasiclassical (for large μ) potential $-3\mu\psi_c(x)^2$. This can be done by the Thomas–Fermi method.[27] Since the spectrum is bounded, $0\leqslant\varepsilon(p)\leqslant J$, for $\mu\gg J$ in the region $|x|<r_c$, where r_c is given by the equation

$$3\mu\psi_c(r_c)^2=J+\kappa^2 \tag{66}$$

states with all p are filled and the electron density is constant and equal to a_0^{-d} for the model (1) with one level per site; for $|x|>r_c$ the electron density drops continuously to zero over a distance r_c. For large $|x|$ the instanton equation (34) can be linearized and expanded in the gradients:[14] The function $\psi_c(x)$ decays exponentially for $\kappa^2>0$ and algebraically for $\kappa^2=0$:

$$\psi_c(x)\propto|x|^{2-d},\quad |x|\to\infty\quad(\kappa^2=0). \tag{67}$$

The desired number of states is of the order of $(r_c/a_0)^d$, and Eqs. (66) and (67) give

$$\mu_s\sim s^{\nu},\quad \nu=(2d-4)/d. \tag{68}$$

For $d>4$ the sequence μ_s increases more rapidly than $s^{1+\delta}$, and substituting Eq. (63) in the form

$$D(z)=\exp\left(-z\sum_s \mu_s^{-1}-\frac{1}{2}z^2\sum_s \mu_s^{-2}-\frac{1}{3}z^3\right.$$
$$\left.\times\sum_s \mu_s^{-3}-\ldots\right) \tag{69}$$

we see that for $d>4$ all sums converge and there are no divergences in $D(z)$; for $d=4$ we have $\mu_s\sim s$ and the first sum in Eq. (69) diverges. Its value is determined by the sum rule[27]

$$\sum_s \frac{1}{\mu_s}=3\int \frac{d^dk}{(2\pi)^d}\frac{1}{\varepsilon(k)+\kappa^2}\sum_x \psi_c(x)^2 \tag{70}$$

obtained by calculating $\ln D(z)$ for small z by perturbation theory based on the definition (61). The integral over k converges due to the discreteness of the model, and the sum over x diverges logarithmically at $d=4$, $\kappa^2\to0$ in accordance with what we have said above; this is an infrared-type divergence, in contrast to the ultraviolet divergence in the continuous model with $d<4$.[27]

Renormalization of the quantity κ eliminates this divergence. Taking κ_0 as the bare value the initial Hamiltonian can be represented as a sum of the Hamiltonian (30) with the renormalized value of κ and the counter term

$$\Delta H\{\varphi\}=\frac{1}{2}\sum_x(\kappa_0^2-\kappa^2)\sum_\alpha \varphi_\alpha(x)^2, \tag{71}$$

where the renormalization of κ is sought in the form of the diagrammatic expansion

$$\kappa_0^2-\kappa^2=C_1g+C_2g^2+\ldots. \tag{72}$$

Constructing the instanton according to the Hamiltonian (30) and estimating the quantities at the saddle point and the rms fluctuations near it as

$$g_c\sim N^{-1},\quad \delta g\sim N^{-3/2},\quad \varphi_c\sim N^{1/2},\quad \delta\varphi\sim1 \tag{73}$$

we prove that in calculations to zeroth order in N inclusively only the first term need be retained in the expansion (72), and the values of g and $\varphi_\alpha(x)$ at the saddle point can be substituted in Eq. (71). Calculating $\kappa_0^2-\kappa^2$ in the one-loop approximation [determining C_1 in Eq. (72)], we find that the counterterm (71) leads to the appearance of an additional factor on the right-hand side of Eq. (60)

$$\exp\left(\frac{1}{2}g_c(n+2)\int\frac{d^dk}{(2\pi)^d}\frac{1}{\varepsilon(k)+\kappa^2}\sum_x\varphi_c(x)^2\right), \tag{74}$$

which, on account of Eqs. (62), (65), (69), and (70), cancels the divergent part of the determinants. Defining the renormalized determinants as

$$D_R(z)=\prod_s\left(1-\frac{z}{\mu_s}\right)\exp\left(\frac{z}{\mu_s}\right), \tag{75}$$

we obtain Eq. (60) in the form

$$[G_M(x_1,\alpha_1,\ldots,x_M,\alpha_M)]_N$$
$$=\frac{2^{n-1}(-1)^N}{(2\pi)^{(n+1)/2}}[|D_R(1)|\bar{D}_R(1/3)^{n-1}]^{-1/2}\Gamma$$
$$\times\Gamma\left(N+\frac{M+n-1}{2}\right)\left(\frac{4}{I_4}\right)^{N+M/2}\sum_y\psi_c(x_1-y)\ldots\psi_c$$
$$\times(x_M-y)\int d^nu\,\delta(|u|-1)u_{\alpha_1}\ldots u_{\alpha_M}, \tag{76}$$

where the instanton solution is determined with the renormalized value of κ.

4.7. κ dependence

The expression (76) is finite at $\kappa=0$ for all $d\geqslant4$. To exhibit the κ-dependence for $\kappa\ll\Lambda$ we add to κ^2 in the instanton equation (34) the increment $\delta\kappa^2$, expand the resulting correction to $\psi_c(x)$ in terms of $e_s^L(x)$, and calculate the change in I_4 using Eq. (52) and the completeness of the basis $e_s^L(x)$. We obtain

$$I_4(\kappa^2+\delta\kappa^2)-I_4(\kappa^2)=2I_2(\kappa^2)\delta\kappa^2. \tag{77}$$

For $d>4$ Eq. (77) can be integrated directly since I_2 is finite at $\kappa^2=0$:

$$I_4(\kappa^2)=I_4(0)+2I_2(0)\kappa^2,\quad d>4. \tag{78}$$

For $d=4$ the quantity $I_2(\kappa^2)$ diverges logarithmically as $\kappa^2\to0$. It is convenient to single out the diverging part by means of the relation, following from Eq. (34), between the Fourier components ψ_c and ψ_c^3

$$\langle\psi_c\rangle_k=\langle\psi_c^3\rangle_k(\varepsilon(k)+\kappa^2)^{-1},\quad \langle f\rangle_k\equiv\sum_x f(x)e^{ikx}. \quad (79)$$

Expressing I_2 in terms of $\langle\psi_c\rangle_k$ and using Eq. (79) we obtain

$$I_2(\kappa^2)=I_3(0)^2K_4\ln\frac{\Lambda}{\kappa},\quad d=4, \quad (80)$$

which, after substitution into Eq. (77) and integration, gives

$$I_4(\kappa^2)=I_4(0)+2I_3(0)^2K_4\kappa^2\ln\frac{\Lambda}{\kappa}+O(\kappa^2),\quad d=4. \quad (81)$$

In the remaining quantities we can set $\kappa=0$ [their κ-dependence corresponds to the energy dependence of the pre-exponential in Eq. (92)].

4.8. Green's function and the self-energy of a disordered system

Setting in Eq. (76) $M=2$ and $\alpha_1=\alpha_2$, Fourier transforming, and passing to the limit $n\to0$ we obtain

$$[G(p)]_N=\frac{(-1)^N}{2(2\pi)^{1/2}}\Gamma\left(N+\frac{1}{2}\right)\times\left(\frac{4}{I_4}\right)^{N+1}\left(\frac{\tilde D_R(1/3)}{|D_R(1)|}\right)^{1/2}\langle\psi_c\rangle_p\langle\psi_c\rangle_{-p}, \quad (82)$$

where we have made use of the fact that

$$\int d^nu\,\delta(|u|-1)u_\alpha^2=\frac{1}{n}\int d^nu\,\delta(|u|-1)|u|^2=\frac{2\pi^{n/2}}{n\Gamma(n/2)}\xrightarrow{n\to0}1. \quad (83)$$

Inverting the series for $G(p)$ (see Appendix) we obtain

$$G(p)^{-1}=(\varepsilon(p)+\kappa_0^2)+\ldots-(\varepsilon(p)+\kappa_0^2)^2[G(p)]_Ng^N-\ldots. \quad (84)$$

Setting $p=0$ in Eq. (84) and using Eq. (9) we obtain the following relation between κ and κ_0:

$$\kappa^2=\kappa_0^2+\ldots-\kappa_0^4[G(0)]_Ng^N-\ldots. \quad (85)$$

Due to the factorial nature of the series (84) and (85), κ can be replaced by κ_0 in the high-order terms of the series (84). Because of Eq. (9) the series (84) gives an expansion for the self-energy. Using Eq. (79) we have

$$[\Sigma(p)]_N=c(p)\Gamma\left(N+\frac{1}{2}\right)\left(-\frac{4}{I_4}\right)^N,$$

$$c(p)=\frac{\langle\psi_c^3\rangle_p\langle\psi_c^3\rangle_{-p}}{(2\pi)^{1/2}}\left(\frac{\tilde D_R(1/3)}{|D_R(1)|}\right)^{1/2}\frac{2}{I_4}. \quad (86)$$

In the limit $p\to0$ we have $\langle\psi_c^3\rangle_p\to I_3$ and the expression (86) remains finite as $p\to0$, $\kappa\to0$. The p and κ dependences are weak; significant changes occur over the distance Λ.

5. STRUCTURE OF THE LEADING APPROXIMATION WITH $d=4$

We set $p=0$ in the expression (86) and expand in powers of κ, using Eq. (81). Comparing to Eq. (13) we identify the coefficients as follows:

$$\Lambda^2B_N=c(0)\Gamma\left(N+\frac{1}{2}\right)\left(-\frac{4}{I_4(0)}\right)^N,$$

$$A_N^0=\text{const}\cdot N\Gamma\left(N+\frac{1}{2}\right)\left(-\frac{4}{I_4(0)}\right)^N,$$

$$A_N^1=-2c(0)\frac{K_4I_3(0)^2}{I_4(0)}N\Gamma\left(N+\frac{1}{2}\right)\left(-\frac{4}{I_4(0)}\right)^N. \quad (87)$$

It is clear from Eq. (87) that the terms with the zeroth and first powers of the logarithms grow most rapidly as $N\to\infty$; the terms with higher powers of the logarithms grow more slowly and are not reproduced by the leading asymptotic relation (86). This identification means that the result (86), derived for $\kappa^2\geq0$, can be extended to arbitrary complex values of κ for $|\kappa|\ll\Lambda$.

We can now formulate the approximation that in the limit of weak disorder gives an asymptotically exact description of the entire energy range, including near Anderson's transition: We include in the expansion (13) (a) the parquet terms, as terms containing the maximum power of a large logarithm, and (b) for $N>N_0$ (with some large N_0) the most rapidly growing terms corresponding to the coefficients (87). The importance of the latter terms is connected with the divergence of the series. For this reason the choice of N_0 is not unimportant. As a result Eq. (13) assumes the form

$$\Sigma(0,\kappa^2)-\text{Re}\,\Sigma(0,0)=\kappa^2\sum_{N=1}^{\infty}A_N^Ng^N\left(\ln\frac{\Lambda}{\kappa}\right)^N+i\,\text{Im}\sum_{N=N_0}^{\infty}c(0)\Gamma\left(N+\frac{1}{2}\right)\times\left(-\frac{4g}{I_4(\kappa^2)}\right)^N. \quad (88)$$

The real part of the last sum can be taken at $\kappa=0$, after which it is included in Re $\Sigma(0,0)$. The quantity κ^2, determined by Eq. (88), is found to be complex: $\kappa^2=-E-i\Gamma$, the damping Γ being exponentially small in the region where the second sum is important. Taking the limit $g\to-W^2/2$ we can set

$$-\frac{4}{I_4(\kappa^2)}g\to\frac{2W^2}{I_4(E)}+i\delta,$$

$$I_4(E)=I_4(0)-I_3(0)^2K_4E\ln\frac{\Lambda^2}{(E^2+\Gamma^2)^{1/2}} \quad (89)$$

and the second sum in Eq. (88) can be calculated using the formula

$$\text{Im}\sum_{N=N_0}^{\infty}\Gamma(N+b)(g+i\delta)^N=\frac{\pi}{g^b}e^{-1/g}, \quad (90)$$

which is obtained by representing the gamma function in the form of its defining integral and summing the resulting geometric progression: This corresponds to Borel summation in the theory of diverging series.[34] Replacing the sum of parquet terms by the result (16), where we set $n=0$, we obtain

$$\Sigma(0,\kappa^2)-\mathrm{Re}\,\Sigma(0,0)=\kappa^2\left[\left(1+8K_4 g\,\ln\frac{\Lambda}{\kappa}\right)^{1/4}-1\right]+i\Gamma_0(0,E), \tag{91}$$

where we have introduced the following notation for the second sum in Eq. (88) with $p\neq 0$:

$$\Gamma_0(p,E)=\pi c(p)\left(\frac{I_4(0)}{2W^2}\right)^{1/2}\exp\left(-\frac{I_4(E)}{2W^2}\right). \tag{92}$$

The E dependence of Γ_0 can be taken into account only in the exponential. This guarantees that Γ_0 and $\nu(E)$ decrease for negative energies over a scale W^2/J; near the Anderson transition, the region of interest to us, $E\sim\Gamma$ and we can set $E=0$ in Eq. (92), after which Eq. (91) differs from the result of the parquette approximation (16) only by the appearance of an imaginary part in $\Sigma(0,0)$: Because of Eq. (11) this substantiates the assertion made in Sec. 1 that the transition "temperature" is complex. Setting $E_c=-\mathrm{Re}\Sigma(0,0)$, denoting the bare energy $-\kappa_0^2$ by E_B (in contrast to the renormalized energy E), and using Eq. (10) we obtain the equation $[\Gamma_0\equiv\Gamma_0(0,E)]$

$$-E_B+E_c=\kappa^2\left(1-4W^2K_4\,\ln\frac{\Lambda}{\kappa}\right)^{1/4}+i\Gamma_0,$$
$$\kappa^2=-E-i\Gamma, \tag{93}$$

which determines E and Γ as functions of E_B.

6. SOLUTION OF EQ. (93)

We set

$$\kappa^2=|\kappa|^2e^{-i\varphi},\quad 1-4W^2K_4\,\ln\frac{\Lambda}{\kappa}=Re^{-i\psi}, \tag{94}$$

where φ and ψ take on values from 0 to π. Taking the imaginary part of Eq. (93) we obtain

$$|\kappa|^2R^{1/4}\sin(\varphi+\psi/4)=\Gamma_0. \tag{95}$$

Setting

$$\Gamma_c=\Lambda^2\exp\left(-\frac{1}{2K_4W^2}\right),\quad x=\ln\frac{|\kappa|^2}{\Gamma_c} \tag{96}$$

we obtain a parametrization of the quantities in Eq. (95) in terms of x and φ

$$R=2K_4W^2(x^2+\varphi^2)^{1/2},\quad \psi=\mathrm{arcctg}\,\frac{x}{\varphi},\quad |\kappa|^2=\Gamma_ce^x, \tag{97}$$

which makes it possible to put Eq. (95) into the form

$$\sin\left(\varphi+\frac{1}{4}\,\mathrm{arcctg}\,\frac{x}{\varphi}\right)=B\,\frac{e^{-x}}{(x^2+\varphi^2)^{1/8}}, \tag{98}$$

where

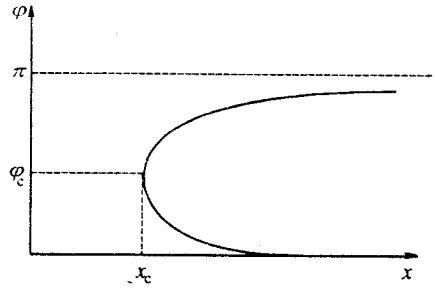

FIG. 2. Plot of the function $\varphi(x)$.

$$B=\frac{\Gamma_0}{\Gamma_c(2K_4W^2)^{1/4}}. \tag{99}$$

Since $\Gamma_0\propto\exp(-S_1)$, where S_1 is assumed to be quite small, we have $B\gg 1$. For $x\lesssim 1$ the right-hand side of Eq. (98) is large for all φ in the interval $(0,\pi)$ and solutions for φ do not exist. Such solutions appear only for $x\gg 1$, when Eq. (98) assumes the form

$$\sin\left(\varphi+\frac{\varphi}{4x}\right)=B\,\frac{e^{-x}}{x^{1/4}} \tag{100}$$

and has the following solutions for φ:

$$\varphi=\begin{cases}\dfrac{4x}{4x+1}\,\arcsin\left(B\,\dfrac{e^{-x}}{x^{1/4}}\right), & \varphi<\varphi_c,\\ \dfrac{4x}{4x+1}\left[\pi-\arcsin\left(B\,\dfrac{e^{-x}}{x^{1/4}}\right)\right], & \varphi>\varphi_c.\end{cases} \tag{101}$$

The function $\varphi(x)$ is displayed in Fig. 2: For $x<x_c$ there are no solutions for φ; for $x>x_c$ there are two solutions, which for large x approach zero and π; the function $x(\varphi)$ is single-valued for all φ in the interval $(0,\pi)$. The parameters x_c and φ_c are determined by the equations

$$\varphi_c=\frac{\pi}{2}\,\frac{4x_c}{4x_c+1},\quad B\,\frac{e^{-x_c}}{x_c^{1/4}}=1 \tag{102}$$

and in the leading approximation are equal to

$$x_c\approx\ln B,\quad \varphi_c\approx\frac{\pi}{2}. \tag{103}$$

The equations

$$E=-\Gamma_ce^x\cos\varphi,\quad \Gamma=\Gamma_ce^x\sin\varphi \tag{104}$$

following from Eq. (94) determine, together with Eq. (101), the function $\Gamma(E)$ in parametric form. For $|E|\gg\Gamma_c$ the following asymptotic relations are valid:

$$\Gamma(E)=\begin{cases}\dfrac{1}{2}\,\pi K_4W^2E[1-2K_4W^2\,\ln(\Lambda^2/E)]^{-1}, & E\gg\Gamma_c,\\ \Gamma_0[1-2K_4W^2\,\ln(\Lambda^2/|E|)]^{-1/4}, & -E\gg\Gamma_c.\end{cases} \tag{105}$$

The one-loop approximation, determining the inverse relaxation time appearing in the kinetic equation, is obtained for large positive E and $\Gamma(E)=\Gamma_0$ holds for large negative E,

i.e., the contribution to the damping is purely nonperturbative. The quantities $|\kappa|^2$ and R reach their minimum values at the point determined by x_c and φ_c:

$$(|\kappa|^2)_{\min}\approx B\Gamma_c=\Gamma_0(2K_4W^2)^{-1/4},$$

$$R_{\min}\approx 2K_4W^2\ln B\approx 1-K_4I_4. \tag{106}$$

To find Γ and E in terms of the bare energy E_B we take the real part of Eq. (93), and substituting Eqs. (94) and (97) and taking into account the fact that $x\gg\varphi$ we obtain

$$-E_B+E_c=\Gamma_c(2K_4W^2)^{1/4}e^x x^{1/4}\cos(\varphi+\varphi/4x), \tag{107}$$

which, together with Eqs. (104) and (101), determines the functions $\Gamma(E_B)$ and $E(E_B)$ in parametric form. For large $|E|$ we have $E\approx E_B-E_c$. The quantity E_c is calculated in the standard manner by perturbation theory and in the one-loop approximation equals

$$E_c=-W^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{\varepsilon(k)}. \tag{108}$$

7. THE SELF-ENERGY $\Sigma(p,\kappa^2)$ AT FINITE MOMENTA

First we calculate $\Sigma(p,\kappa^2)$ in the parquet approximation. This can be done by extending somewhat Ginzburg's work.[21] From the Ward identities we obtain

$$\frac{\partial G^{-1}(p,\kappa^2)}{\partial\kappa_0^2}\delta_{\alpha\beta}=\delta_{\alpha\beta}-\frac{1}{2}\sum_\sigma\int\frac{d^dq}{(2\pi)^d}G(q)^2\times$$

$$\times P_{\alpha\sigma\sigma\beta}(p,-q,q,-p), \tag{109}$$

where $P_{\alpha\beta\mu\nu}(p_1,p_2,p_3,p_4)$ is the total four-leg vertex $(p_1+p_2+p_3+p_4=0)$. For $d=4$ the integral is logarithmic since $G(q)\sim q^{-2}$. Following the usual procedure employed in parquet calculations[20,21] we transform at the vertex from the momenta p_1, p_2, p_3, and p_4 to the momenta p', p, and q, where

$$2p'=p_4-p_3,\quad 2p=p_1-p_2,\quad q=p_1+p_2=-p_3-p_4 \tag{110}$$

and the tips of the four legs are labelled so that $p'>p>q>0$. Introducing the logarithmic variables

$$y=\ln\frac{\Lambda}{p},\quad z=\ln\frac{\Lambda}{q},\quad z_\infty=\ln\frac{\Lambda}{\kappa} \tag{111}$$

we rewrite Eq. (109) in the form

$$\frac{\partial G^{-1}(y)}{\partial\kappa_0^2}\delta_{\alpha\beta}=\delta_{\alpha\beta}-\frac{K_4}{2}\sum_\sigma\left(\int_0^y dzP_{\alpha\beta\sigma\sigma}(z,y,z_\infty)\right.$$

$$\left.+\int_y^{z_\infty}dzP_{\sigma\sigma\beta\alpha}(y,z,z_\infty)\right), \tag{112}$$

where we have taken into account the fact that in parquet calculations it is assumed that $q\gg\kappa$ and for this reason the limit $q\to 0$ is interpreted as $q\to\kappa$. Using the fact that P has the structure[21]

$$P_{\alpha\beta\mu\nu}(x,y,z)=T_1(x,y,z)I_{\alpha\beta\mu\nu}+T_2(x,y,z)\delta_{\alpha\beta}\delta_{\mu\nu},$$

$$I_{\alpha\beta\mu\nu}=\delta_{\alpha\beta}\delta_{\mu\nu}+\delta_{\alpha\mu}\delta_{\beta\nu}+\delta_{\alpha\nu}\delta_{\beta\mu} \tag{113}$$

and substituting for T_1 and T_2 the expressions (21) and (22) from Ref. 21 we obtain

$$\frac{\partial G^{-1}(y,\kappa^2)}{\partial\kappa_0^2}=-\frac{6}{n-4}t(y)^{-(n+2)/(n+8)}$$

$$+\frac{n+2}{n-4}t(y)^{-6/(n+8)}t(z_\infty)^{-(4-n)/(n+8)}$$

$$t(y)=1+K_4(n+8)gy, \tag{114}$$

which agrees with Eq. (15) in the limit $p\to 0$ (when $y\to z_\infty$). Dividing Eq. (114) by Eq. (15), we find $\partial G^{-1}/\partial\kappa^2$; integrating over κ^2 in the leading logarithmic approximation, determining the integration constant according to Eq. (9), and dropping the quantity $\Sigma(p,0)-\Sigma(0,0)$, a contribution to which arises only in the second logarithmic approximation,[21] we obtain the desired parquet approximation for $\Sigma(p,\kappa^2)$:

$$\Sigma(p,\kappa^2)=\Sigma(0,0)+\kappa^2\left(t(z_\infty)^{(n+2)/(n+8)}\right.$$

$$+\frac{6}{n-4}\left[\frac{t(z_\infty)}{t(y)}\right]^{(n+2)/(n+8)}$$

$$\left.-\frac{n+2}{n-4}\left[\frac{t(z_\infty)}{t(y)}\right]^{6/(n+8)}\right). \tag{115}$$

As in the case $p=0$, the contribution of the most rapidly growing terms in the perturbation series gives rise to an additional term $i\Gamma_0(p,E)$ [see Eq. (92)] on the right-hand side of Eq. (115). Passing to the limit $n\to 0$ we obtain the final result

$$\Sigma(p,\kappa^2)-\Sigma(0,\kappa^2)=\kappa^2\left(1-\frac{3}{2}\left[\frac{t(z_\infty)}{t(y)}\right]^{1/4}\right.$$

$$\left.+\frac{1}{2}\left[\frac{t(z_\infty)}{t(y)}\right]^{3/4}\right)+i\Gamma_0(p,E)$$

$$-i\Gamma_0(0,E). \tag{116}$$

8. CALCULATION OF THE DENSITY OF STATES

The density of states $\nu(E)$ is determined by the formula

$$\nu(E)=\frac{1}{\pi}\operatorname{Im}\int\frac{d^4p}{(2\pi)^4}G(p,\kappa^2)$$

$$\equiv\frac{1}{\pi}\operatorname{Im}Y(\kappa^2)|_{\kappa^2=-E-i\Gamma}. \tag{117}$$

The region $|p|\gtrsim\kappa$, where

$$Y(\kappa^2)\approx\int\frac{d^4p}{(2\pi)^4}\frac{1}{\varepsilon(p)}$$

$$-\int\frac{d^4p}{(2\pi)^4}\frac{\kappa^2-\Sigma(p,\kappa^2)+\Sigma(0,\kappa^2)}{\varepsilon(p)^2}, \tag{118}$$

makes the main contribution to the integral. Substituting the expression (116) into Eq. (118) we can drop the difference of the nonperturbative contributions—no logarithmic situation occurs

here. This reduces the power of the logarithm by one. Switching in Eq. (118) to logarithmic variables and integrating we obtain

$$\nu(E)=\frac{1}{2\pi W^2}\,\mathrm{Im}\,\kappa^2\Bigg(-\left[1-4K_4W^2\ln\frac{\Lambda}{\kappa}\right]^{1/4}+\left[1-4K_4W^2\ln\frac{\Lambda}{\kappa}\right]^{3/4}\Bigg)_{\kappa^2=-E-i\Gamma}. \tag{119}$$

Using Eqs. (94) and (97) and $x\gg\varphi$ gives

$$\nu(E)=\frac{\Gamma_c e^x}{2\pi W^2}(2K_4W^2x)^{1/4}\left[\sin\left(\varphi+\frac{\varphi}{4x}\right)-(2K_4W^2x)^{1/2}\sin\left(\varphi+\frac{3\varphi}{4x}\right)\right], \tag{120}$$

which together with Eqs. (101) and (104) determines $\nu(E)$ parametrically. For $|E|\gg\Gamma_c$ we have the asymptotic relations

$$\nu(E)=\begin{cases}\dfrac{1}{2}K_4E\left(1-2K_4W^2\ln\dfrac{\Lambda^2}{E}\right)^{-1/4}, & E\gg\Gamma_c,\\ \Gamma_0(2\pi W^2)^{-1}\left[1-\left(1-2K_4W^2\ln\dfrac{\Lambda^2}{|E|}\right)^{1/2}\right], & -E\gg\Gamma_c.\end{cases} \tag{121}$$

For large positive E $\nu(E)$ becomes the density of states of an ideal lattice, and for large negative E, using Eq. (92), we obtain the result

$$\nu(E)=\frac{1}{2}K_4c(0)\ln\frac{\Lambda^2}{|E|}\left(\frac{I_4(0)}{2W^2}\right)^{1/2}\exp\left(-\frac{I_4(E)}{2W^2}\right), \tag{122}$$

which can be found by summing the higher-order terms of the series for $G(p)$ [see Eq. (82)]: it corresponds to the asymptotic behavior obtained for the fluctuation-induced tail by the instanton method in the usual formulation.[25,28,29] The density of states $\nu(E)$ given by Eq. (120) has no singularities for any values of E.

9. ESTIMATE OF CORRECTIONS TO THE LEADING APPROXIMATION

We now show that the corrections to the approximation employed are indeed small. These corrections can be divided into two groups.

On the one hand, corrections $\sim 1/N$, $1/N^2$, and so on to the leading-order asymptotic expression of the expansion coefficients in the Lipatov method, which arise when higher powers of δg and $\delta\varphi$ are included together with the higher-order terms in the expansion (72), can be taken into account. Then expressions of the type (86) contain the extra factor $1/N^m$, which is equivalent decreasing the argument of the gamma function by m, which on summing according to Eq. (90) gives the extra factor $\sim g^m\ll 1$. Although higher powers of the logarithms then arise, these powers appear in the combination $\kappa^2\ln^k(\Lambda/\kappa)$ and in each order in $1/N$ they are small compared to the term $\sim\Lambda^2$, containing the zeroth power of the logarithm.

On the other hand, corrections to the parquet approximation can be taken into account by including the coefficients A_N^{N-1}, A_N^{N-2}, and so on: They grow more rapidly than A_N^N, and they can be significant. Because of Eq. (19) the summation in Eq. (13) of terms with $K=N-m$ gives a contribution to Σ

$$\sim\kappa^2 g^m\left(1+8K_4g\ln\frac{\Lambda}{\kappa}\right)^{1/4-m}\ln^m\left(1+8K_4g\ln\frac{\Lambda}{\kappa}\right), \tag{123}$$

whose ratio to the parquet result has the upper limit

$$\sim(g(\ln R_{\min})/R_{\min})^m \tag{124}$$

since the quantity R in Eq. (94) lies in the range from $R_{\min}$ to 1. Since I_4 is small for a strongly attractive core, the expression (106) gives $R_{\min}\sim 1$, which means that the expression (124) is small. As $K_4I_4\to 1$ we have $R_{\min}\to 0$ from Eq. (106) and the corrections (124) become significant. On the other hand, for $K_4I_4>1$ Eq. (98) has no physically satisfactory solutions, since it does not determine a single-valued function $x(\varphi)$ for all φ. This indicates the existence of a critical value of the instanton minimum S_1 (equal to 3/4 of the value of S_0 for the plateau) up to which the theory expounded above is valid. The question of whether or not this critical value is meaningful or is eliminated when the contribution of the instanton plateau is taken into account (Sec. 3) requires further investigation.

10. REFINEMENT OF THE RESULTS FOR $d>4$

The results of Sec. 4 determine the coefficient c in the expression (92) for the nonperturbative contribution Γ_0 and its momentum dependence. This makes it possible to refine the results of the preceding work (Ref. 14) for $d>4$. For $d>4$ the first term and the sum of the remote terms of the series are retained [Ref. 14] in the expansion of $\Sigma(p,\kappa^2)$ in powers of g; this gives in terms of the bare energy E_B

$$\Sigma(p,E_B)=W^2\int\frac{d^dk}{(2\pi)^d}\frac{1}{\varepsilon(k)-E_B-\Sigma(\kappa,E_B)}+i\Gamma_0(p,E_B). \tag{125}$$

The real part of Σ does not depend on p and it is possible to switch to the renormalized energy

$$E=E_B+\mathrm{Re}\,\Sigma(E_B)\approx E_B-E_c,\qquad E_c=-W^2\int\frac{d^dk}{(2\pi)^d}\frac{1}{\varepsilon(k)}. \tag{126}$$

Then

$$\Gamma(p,E)=\mathrm{Im}\,\Sigma(p,E)=\pi W^2\nu(E)+\Gamma_0(p,E) \tag{127}$$

and the following equation is obtained for the density of states $\nu(E)$:

$$\nu(E)=\frac{1}{\pi}\,\mathrm{Im}$$

$$\times\int \frac{d^dk}{(2\pi)^d}\frac{1}{\varepsilon(k)-E-i\pi W^2\nu(E)-i\Gamma_0(k,E)}, \tag{128}$$

where $\Gamma_0(k,E)$ is determined by Eqs. (92) and (78) with $\kappa^2=-E$.

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APPENDIX. OPERATIONS WITH FACTORIAL SERIES.

Let

$$S_A=A_0+A_1g+\ldots+A_Ng^N+\ldots,$$
$$S_B=B_0+B_1g+\ldots+B_Ng^N+\ldots \tag{A1}$$

be two factorial series, so that $A_N, B_N\sim N!$ Multiplying the series we have

$$S_AS_B=A_0B_0+\ldots+g^N(A_0B_N+A_1B_{N-1}+A_2B_{N-2}+\ldots$$
$$+A_NB_0)+\ldots. \tag{A2}$$

In view of the factorial nature of the series $B_{N-1}\sim B_n/N$, $B_{N-2}\sim B_N/N^2$, and so on, and similarly $A_{N-1}\sim A_N/N$, $A_{N-2}\sim A_N/N^2$, and so on. For this reason only the first and last terms need be retained in the parentheses in Eq. (A2):

$$S_AS_B=A_0B_0+\ldots+g^N(A_0B_N+A_NB_0)+\ldots, \tag{A3}$$

if the coefficients A_N and B_N grow at the same rate. If the coefficients of one of the series, for example S_A, grow more rapidly (due to the more slowly increasing corrections to the principal factorial function), then only the second term remains in the parentheses in Eq. (A3) and the product S_AS_B can be written as B_0S_A, if only the zeroth and higher order terms in the expansion are taken into account.

Using Eq. (A3) to multiply the series S_A n times by itself we obtain the following equation for the nth power of S_A:

$$(S_A)^n=A_0^n+\ldots+nA_0^{n-1}A_Ng^N+\ldots. \tag{A4}$$

To calculate a negative or fractional power of S_A we employ in the expression

$$(S_A)^\alpha=A_0^\alpha[1+g(\tilde{A}_0+\ldots+\tilde{A}_Ng^N+\ldots)]^\alpha,$$
$$\tilde{A}_n=A_{n+1}/A_0$$

the series expansion of $(1+x)^\alpha$ and we apply the result (A4) to the factorial series in parentheses:

$$(S_A)^\alpha=A_0^\alpha\left[1+\ldots+g^N\left(\alpha\tilde{A}_{N-1}+\frac{\alpha(\alpha-1)}{2!}\right.\right.$$
$$\left.\left.\times2\tilde{A}_0\tilde{A}_{N-2}+\ldots\right)+\ldots\right]. \tag{A5}$$

Retaining in parentheses only the first term we obtain the extension of Eq. (A4) to arbitrary real values of n.

If the function $f(x)$ is given by the converging series

$$f(x)=f_0+f_1x+\ldots+f_Nx^N+\ldots \tag{A6}$$

then substituting $x=S_A$ and using Eq. (A4), we obtain

$$f(S_A)=f(A_0)+\ldots+g^Nf'(A_0)A_N+\ldots. \tag{A7}$$

[1)] In the diagrammatic technique the fact that g is negative is not significant because the expansion is made in integer powers of g. Functional integrals with $g<0$ are understood in the sense of an analytic continuation from positive g. Actually, as explained in Ref. 15, when using the replica method the appearance of diverging functional integrals can be avoided by exploiting the arbitrariness in the choice of the field φ (the coefficients in the Hamiltonian (4) then become complex). This procedure does not, however, eliminate the difficulties associated with the appearance of a spurious pole, which are described below.

[2)] In the estimates given below constants of order ~1 are dropped.

[3)] In what follows, whenever a choice is important, we have in mind the retarded Green's function.

[4)] The fact that λ_s^L with $s\neq0$ is positive is obvious from the following: In the continuous model, when the instanton radius is large, the zeroth modes of the operators $\hat{M}_T$ and $\hat{M}_L$ are associated with the fact that the potential $-|g_c|\varphi_c(x)^2$ has a level $-\kappa^2$ corresponding to the angular momentum $l=0$ and the potential $-3|g_c|\varphi_c(x)^2$ contains a d-fold degenerate level corresponding to $l=1$ (the deepening of the potential is compensated by the centrifugal energy).[25] As the instanton radius decreases, states with $l=1$ strive to break through into the continuous spectrum, since the delta-function-like potential has only one level.

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