

Analytical asymptotics of β -function in φ^4 theory (end of the "zero charge" story)

I. M. Suslov

P.L.Kapitza Institute for Physical Problems,
119337 Moscow, Russia

Abstract

Reconstruction of the β -function for φ^4 theory, attempted previously by summation of perturbation series, led to the asymptotics $\beta(g) = \beta_\infty g^\alpha$ at $g \rightarrow \infty$, where $\alpha \approx 1$ for space dimensions $d = 2, 3, 4$. The natural hypothesis arises, that asymptotic behavior is $\beta(g) \sim g$ for all d . Consideration of the "toy" zero-dimensional model confirms the hypothesis and reveals the origin of this result: it is related with a zero of a certain functional integral. Consideration can be generalized to the arbitrary space dimensionality, confirming the linear asymptotics of $\beta(g)$ for all d . Asymptotical behavior for other renormalization group functions (anomalous dimensions) is found to be constant. Relation to the "zero charge" problem is discussed.

1. Introduction

According to Landau, Abrikosov, Khalatnikov [1], relation of the bare charge g_0 with observable charge g for renormalizable field theories is given by expression

$$g = \frac{g_0}{1 + \beta_2 g_0 \ln \Lambda/m}, \quad (1)$$

where m is the mass of the particle, and Λ is the momentum cut-off. For finite g_0 and $\Lambda \rightarrow \infty$ the "zero charge" situation ($g \rightarrow 0$) takes place. The proper interpretation of Eq.1 consists in its inverting, so that g_0 (related to the length scale Λ^{-1}) is chosen to give a correct value of g :

$$g_0 = \frac{g}{1 - \beta_2 g \ln \Lambda/m}. \quad (2)$$

The growth of g_0 with Λ invalidates Eqs.1,2 in the region $g_0 \sim 1$ and existence of "the Landau pole" in Eq.2 has no physical sense.

The actual behavior of the charge $g(L)$ as a function of the length scale L is determined by the Gell-Mann – Low equation

$$-\frac{dg}{d \ln L} = \beta(g) = \beta_2 g^2 + \beta_3 g^3 + \dots \quad (3)$$

and depends on the appearance of the function $\beta(g)$. According to classification by Bogolyubov and Shirkov [2], the growth of $g(L)$ is saturated, if $\beta(g)$ has a zero for finite g , and continues to infinity, if $\beta(g)$ is non-alternating and behaves as $\beta(g) \sim g^\alpha$ with $\alpha \leq 1$ for large g ; if, however, $\beta(g) \sim g^\alpha$ with $\alpha > 1$, then $g(L)$ is divergent at finite $L = L_0$ (the real Landau pole arises) and the theory is internally inconsistent due to indeterminacy of $g(L)$ for $L < L_0$. The latter case corresponds to the "zero charge" situation in full theory (beyond its perturbation framework).

One can see that solution of the "zero charge" problem needs calculation of the Gell-Mann – Low function $\beta(g)$ at arbitrary g , and in particular its asymptotic behavior for $g \rightarrow \infty$. Such attempt was made recently by the present author for φ^4 theory [3], QED [4] and QCD [5] (see a review article [6]). It is based on the fact that the first four coefficients β_N in Eq.3 are known from diagrammatic calculations, while their large order behavior can be established by the Lipatov method [7, 6]. Smooth interpolation of the coefficient function and the proper summation of the perturbation series give non-alternating $\beta(g)$ with $\alpha \approx 1$ in four-dimensional φ^4 theory [3]¹. Recent results for 2D and 3D φ^4 theory [12, 13] also correspond to $\alpha \approx 1$. The natural hypothesis arises, that $\beta(g)$ has the linear asymptotics for arbitrary space dimension d . Simplicity of the result indicates that it can be obtained analytically.

Below we show that it is so indeed. Analysis of zero-dimensional theory confirms the asymptotics $\beta(g) \sim g$ and reveals its origin. It is related with unexpected circumstance that the strong coupling limit for the renormalized charge g is determined not by large values of the bare charge g_0 , but its complex values.² More than that, it is sufficient to consider the region $|g_0| \ll 1$, where the functional integrals can be evaluated in the saddle-point approximation. If a proper direction in the complex g_0 plane is chosen, the saddle-point contribution of the trivial vacuum is comparable with the saddle-point contribution of the main instanton, and a functional integral can turn to zero. The limit $g \rightarrow \infty$ is related with the zero of a certain functional integral and appears to be completely controllable. As a result, it is possible to obtain asymptotic behavior of the β -function and anomalous dimensions: the former indeed appears to be linear, while the latter achieve certain constant limits. Analogous results can be obtained for QED [17].

The asymptotics $\beta(g) \sim g$ in combination with non-alternating behavior of $\beta(g)$ corresponds to the second possibility in the Bogolyubov–Shirkov classification: $g(L)$ is finite for large L but grows to infinity at $L \rightarrow 0$. It looks in conflict with the expected triviality of φ^4 theory (see e.g. [14] and the references therein). In fact, two definitions of triviality were mixed in the literature. The first one, introduced by Wilson [15], is equivalent to positiveness of $\beta(g)$ for $g \neq 0$; it is confirmed by all available information and can be

¹ Possibility of correct summation of perturbation series is frequently questioned in relation with possible existence of renormalon singularities in the Borel plane [8]. Such singularities can be easily obtained by summing some special sequences of diagrams, but their existence was never proved, if all diagrams are taken into account [9]. The present result for the asymptotics of $\beta(g)$ satisfies the general criterion for absence of renormalon singularities [10] and confirms the proof of their absence suggested in [11].

² Discussion of unitarity of theory is given in Sec.5.

considered as firmly established. The second definition, introduced by mathematical community [16], corresponds to the true triviality and is equivalent to internal inconsistency in the Bogolyubov–Shirkov sense: it needs not only positiveness of $\beta(g)$ but also the corresponding asymptotical behavior. Evidence of true triviality is not extensive and allows different interpretation [3]. The present analysis gives new insight to this problem: to obtain nontrivial theory one need to use the complex values of the bare charge g_0 , which were never exploited in mathematical proofs and numerical simulations. This matters will be discussed in a separate paper [18].

2. Definition of the renormalization group (RG) functions

Consider the $O(n)$ symmetric φ^4 theory with an action

$$S\{\varphi\} = \int d^d x \left\{ \frac{1}{2} \sum_{\alpha} (\nabla \varphi_{\alpha})^2 + \frac{1}{2} m_0^2 \sum_{\alpha} \varphi_{\alpha}^2 + \frac{1}{8} u \left(\sum_{\alpha} \varphi_{\alpha}^2 \right)^2 \right\},$$

$$u = g_0 \Lambda^{\epsilon}, \quad \epsilon = 4 - d \quad (4)$$

in d -dimensional space. Following the usual RG formalism [19], consider the "amputated" vertex $\Gamma^{(L,N)}$ with N external lines of field φ and L insertions of φ^2 (where two fields φ are taken in the coinciding spatial points). Its multiplicative renormalizability means

$$\Gamma^{(L,N)}(p_i; g_0, m_0, \Lambda) = Z^{-N/2} \left(\frac{Z_2}{Z} \right)^{-L} \Gamma_R^{(L,N)}(p_i; g, m), \quad (5)$$

i.e. divergency at $\Lambda \rightarrow \infty$ disappears after extracting the proper Z -factors and transferring to the renormalized charge and mass. We accept the renormalization conditions at zero momentum³

$$\begin{aligned} \Gamma_R^{(0,2)}(p; g, m) \Big|_{p \rightarrow 0} &= m^2 + p^2 + O(p^4), \\ \Gamma_R^{(0,4)}(p_i; g, m) \Big|_{p_i=0} &= gm^{\epsilon}, \\ \Gamma_R^{(1,2)}(p_i; g, m) \Big|_{p_i=0} &= 1, \end{aligned} \quad (6)$$

which are typical for applications in the phase transitions theory [20]. Substitution of (6) into (5) gives expressions for g, m, Z, Z_2 in terms of the bare quantities

$$\begin{aligned} Z(g_0, m_0, \Lambda) &= \left(\frac{\partial}{\partial p^2} \Gamma^{(0,2)}(p; g_0, m_0, \Lambda) \Big|_{p=0} \right)^{-1}, \\ Z_2(g_0, m_0, \Lambda) &= \left(\Gamma^{(1,2)}(p_i; g_0, m_0, \Lambda) \Big|_{p_i=0} \right)^{-1}, \end{aligned} \quad (7)$$

³ Dependence on the renormalization scheme is discussed in Sec.5.

$$m^2 = Z(g_0, m_0, \Lambda) \Gamma^{(0,2)}(p; g_0, m_0, \Lambda) \Big|_{p=0},$$

$$g = m^{-\epsilon} Z^2(g_0, m_0, \Lambda) \Gamma^{(0,4)}(p_i; g_0, m_0, \Lambda) \Big|_{p_i=0}.$$

Applying differential operator $d/d \ln m$ to (5) for fixed g_0 and Λ gives the Callan-Symanzik equation, valid asymptotically for large p_i/m [19]

$$\left[\frac{\partial}{\partial \ln m} + \beta(g) \frac{\partial}{\partial g} + (L - N/2) \eta(g) - L \eta_2(g) \right] \Gamma_R^{(L,N)}(p_i; g, m) \approx 0, \quad (8)$$

where the RG functions $\beta(g)$, $\eta(g)$ and $\eta_2(g)$ are determined as

$$\beta(g) = \frac{dg}{d \ln m} \Big|_{g_0, \Lambda = const}, \quad \eta(g) = \frac{d \ln Z}{d \ln m} \Big|_{g_0, \Lambda = const}, \quad \eta_2(g) = \frac{d \ln Z_2}{d \ln m} \Big|_{g_0, \Lambda = const} \quad (9)$$

and according to general theorems depend only on g [19].

3. "Naive" zero-dimensional limit.

The functional integrals of φ^4 theory are determined as

$$Z_{\alpha_1 \dots \alpha_M}^{(M)}(x_1, \dots, x_M) = \int D\varphi \varphi_{\alpha_1}(x_1) \varphi_{\alpha_2}(x_2) \dots \varphi_{\alpha_M}(x_M) \exp(-S\{\varphi\}). \quad (10)$$

To take a zero-dimensional limit, consider the system restricted spatially in all directions at sufficiently small scale, and neglecting spatial dependence of $\varphi(x)$ omit the terms with gradients in Eq.4; interpreting the functional integral as a multi-dimensional integral on a lattice, we can take the system sufficiently small, so it contains only one lattice site:

$$Z_{\alpha_1 \dots \alpha_M}^{(M)} = \int d^n \varphi \varphi_{\alpha_1} \dots \varphi_{\alpha_M} \exp\left(-\frac{1}{2} m_0^2 \varphi^2 - \frac{1}{8} u \varphi^4\right). \quad (11)$$

The diagrammatic expansions generated by such "functional" integrals have the usual form, but all propagators should be taken at zero momenta and no momentum integrations are necessary.

Such understanding of zero-dimensional theory is conventional in the literature. However, it does not quite correspond to the true zero-dimensional limit of φ^4 theory. Considering expressions for the simplest diagrams in d -dimensional case and taking limit $d \rightarrow 0$, it is easy to be convinced that their trivialization (of the described type) occurs only for zero external momenta; if the latter are different from zero, no evident simplifications occur. This point is essential for definition of the Z -factor, which is introduced according to a scheme (see the first relation in (6))

$$G_2(p) = \frac{1}{p^2 + m_0^2 + \Sigma(p, m_0)} = \frac{1}{p^2 + m_0^2 + a_0(m_0) + a_2(m_0)p^2 + a_4(m_0)p^4 + \dots} =$$

$$= \frac{Z}{p^2 + m^2 + O(p^4)}, \quad (12)$$

and is determined by the momentum dependence of self-energy. In the described "naive" zero-dimensional theory, non-zero momenta are absent and we can accept $Z = 1$. Such procedure is internally consistent but does not correspond to the true zero-dimensional limit of φ^4 theory. The latter fact is not essential for us, since this model is used only for illustration and the proper consideration of the general d -dimensional case will be given in the next section.

Substituting $\varphi_\alpha = \varphi u_\alpha$ in (11) and integrating over directions of the unit vector \mathbf{u} , we obtain for even M [21]

$$Z_{\alpha_1 \dots \alpha_M}^{(M)} = \frac{2\pi^{n/2}}{2^{M/2} \Gamma(M/2 + n/2)} I_{\alpha_1 \dots \alpha_M} K_M(m_0, u), \quad (13)$$

where $I_{\alpha_1 \dots \alpha_M}$ is the sum of terms like $\delta_{\alpha_1 \alpha_2} \delta_{\alpha_3 \alpha_4} \dots$ with all possible pairings, and

$$K_M(m_0, u) = \int_0^\infty \varphi^{M+n-1} d\varphi \exp\left(-\frac{1}{2}m_0^2 \varphi^2 - \frac{1}{8}u\varphi^4\right). \quad (14)$$

Defining the M -point Green functions $G^{(M)}$ as $Z^{(M)}/Z^{(0)}$ and extracting dependence on indices

$$G_{\alpha\beta}^{(2)} = G_2 \delta_{\alpha\beta}, \quad G_{\alpha\beta\gamma\delta}^{(4)} = G_4 I_{\alpha\beta\gamma\delta}, \quad \Gamma_{\alpha\beta\gamma\delta}^{(0,4)} = \Gamma_4 I_{\alpha\beta\gamma\delta}, \quad (15)$$

we have

$$\Gamma_2 = 1/G_2, \quad G_4 = G_2^2 - G_4^2 \Gamma_4, \quad (16)$$

where

$$G_2 = \frac{1}{n} \frac{K_2(m_0, u)}{K_0(m_0, u)}, \quad G_4 = \frac{1}{n(n+2)} \frac{K_4(m_0, u)}{K_0(m_0, u)} \quad (17)$$

and the vertex $\Gamma_{\alpha\beta\gamma\delta}^{(0,4)}$ is defined by the usual relation

$$G_{\alpha\beta\gamma\delta}^{(4)} = G_{\alpha\beta}^{(2)} G_{\gamma\delta}^{(2)} + G_{\alpha\gamma}^{(2)} G_{\beta\delta}^{(2)} + G_{\alpha\delta}^{(2)} G_{\beta\gamma}^{(2)} - G_{\alpha\alpha'}^{(2)} G_{\beta\beta'}^{(2)} G_{\gamma\gamma'}^{(2)} G_{\delta\delta'}^{(2)} \Gamma_{\alpha'\beta'\gamma'\delta'}^{(0,4)}. \quad (18)$$

Using the renormalization conditions (7), we obtain

$$m^2 = \Gamma_2 = \frac{nK_0}{K_2}, \quad (19)$$

$$g = \frac{\Gamma_4}{m^4} = 1 - m^4 G_4 = 1 - \frac{n}{n+2} \frac{K_4 K_0}{K_2^2}. \quad (20)$$

Differentiating (19) over m_0^2 and taking into account that differentiation of K_M transfers it to K_{M+2} (see (14)), we have

$$\frac{dm^2}{dm_0^2} = \frac{n}{2} \left\{ -1 + \frac{K_4 K_0}{K_2^2} \right\}. \quad (21)$$

Since all differentiations in (9) occur at g_0 , $\Lambda = \text{const}$, the latter parameters are considered to be fixed throughout all calculations: then m^2 is a function of only m_0^2 and Eq.21 defines also the derivative dm_0^2/dm^2 . According to definition of the β -function (9) we have

$$\beta(g) = 2 \frac{dg}{d \ln m^2} = -\frac{2m^4}{n(n+2)} \left[2 \frac{K_4}{K_0} + \left(\frac{K_4}{K_0} \right)'_{m_0^2} m^2 \frac{dm_0^2}{dm^2} \right] \quad (22)$$

and substitution of (21) gives the following expression

$$\beta(g) = -\frac{2n}{n+2} \frac{K_4 K_0}{K_2^2} \left[2 + \frac{\frac{K_6 K_0}{K_4 K_2} - 1}{1 - \frac{K_4 K_0}{K_2^2}} \right]. \quad (23)$$

The change of variables $\varphi \rightarrow \varphi(8/u)^{1/4}$ in the integrals (14) reduces them to the form

$$K_M(t) = \int_0^\infty \varphi^{M+n-1} d\varphi \exp(-t\varphi^2 - \varphi^4), \quad t = \left(\frac{2}{u} \right)^{1/2} m_0^2. \quad (24)$$

The arising factors drop out of the combinations $K_4 K_0 / K_2^2$ and $K_6 K_0 / K_4 K_2$ entering equations (20),(23), and the latter have the same form in terms of $K_M(t)$, as they had in terms of $K_M(m_0, u)$. The right hand sides of (20),(23) are the functions of the single variable t and the dependence $\beta(g)$ is determined by these expressions in the parametric form.

The vertex $\Gamma_{\alpha\beta}^{(1,2)} = \Gamma_{12} \delta_{\alpha\beta}$ is determined by the Ward identity,

$$\Gamma_{12} = \frac{dm^2}{dm_0^2} = 1 - \frac{n+2}{2} g, \quad (25)$$

and the function $\eta_2(g)$ is given by expression

$$\eta_2(g) = -\frac{d \ln \Gamma_{12}}{d \ln m} = \frac{\beta(g)}{2/(n+2) - g}, \quad (26)$$

while $\eta(g)$ is identically zero in the accepted approximation.

Using the asymptotic expressions for $K_M(t)$,

$$K_M(t) = \begin{cases} \frac{1}{\sqrt{2}} t^{-(M+n)/2} \Gamma\left(\frac{M+n}{2}\right) \left[1 - \frac{(M+n)(M+n+2)}{4t^2} + \dots \right], & t \rightarrow \infty \\ \frac{1}{4} \left[\Gamma\left(\frac{M+n}{4}\right) - t \Gamma\left(\frac{M+n+2}{4}\right) + \dots \right], & t \rightarrow 0 \\ \frac{\sqrt{\pi}}{2} e^{t^2/4} \left(\frac{|t|}{2}\right)^{(M+n-2)/2} \left[1 + \frac{(M+n-2)(M+n-4)}{4t^2} + \dots \right], & t \rightarrow -\infty, \end{cases} \quad (27)$$

it is easy to obtain that g and $\beta(g)$ depend on t as shown in Fig.1,a, i.e. variation of

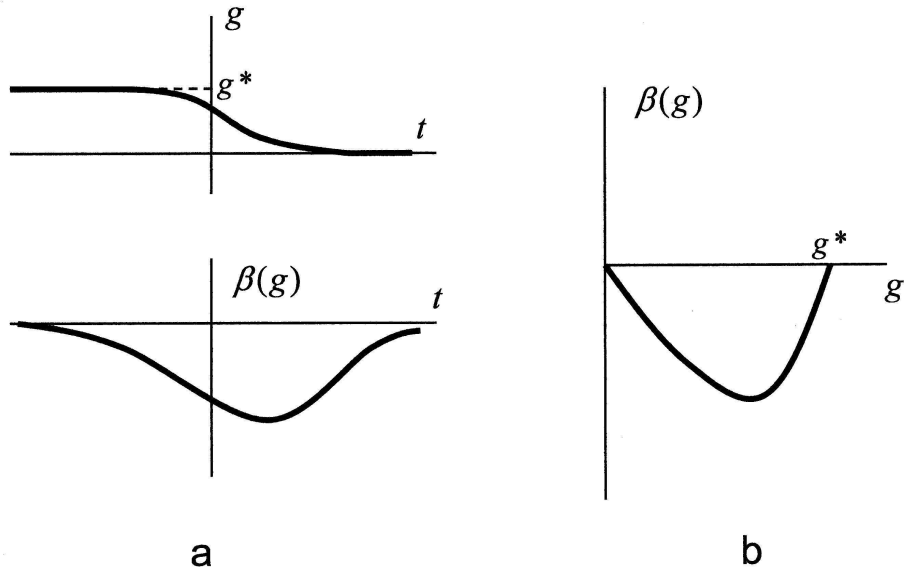


Figure 1: (a) Dependence of g and $\beta(g)$ on the parameter t . (b) Resulting appearance of $\beta(g)$.

parameter t along the real axis determines $\beta(g)$ in the interval from $g = 0$ till the fixed point (Fig. 1,b)⁴

$$g^* = \frac{2}{n+2}. \quad (28)$$

To advance into the large g region, one should investigate the parametric representation (20), (23) for complex values of t . If $t = |t|e^{i\chi}$ and $|t| \gg 1$, then (in dependence on χ) the integrals $K_M(t)$ are determined by either trivial saddle point $\varphi = 0$, or nontrivial saddle-point $\varphi^2 = -t/2$. The saddle-point contributions to $K_M(t)$ depend on t , but this dependence drops out of the combinations K_4K_0/K_2^2 and K_6K_0/K_4K_2 , entering (20,23). Thus, in the rough approximation, the complex t plane is divided into two parts where g and $\beta(g)$ takes constant values $g = 0$, $\beta(g) = 0$ and $g = g^*$, $\beta(g) = 0$. The smooth transition between these values is related with deviations from the saddle-point approximation, which arise for $|t| \lesssim 1$; however, corresponding variations of g are expected to be finite, as in the case of the real t (Fig. 1,a). Now it is easy to understand that large values of g can be achieved only in those directions of the complex t plane where contributions from two

⁴Existence of the fixed point g^* does not mean the existence of the phase transition, which is absent for $d < 2$. The scaling behavior of correlation functions follows from the Callan–Symanzik equation only in the region of small m , which is inaccessible for physical values of m_0 and g_0 . Eq.(28) is in agreement with the result $\tilde{g}^* = (n+8)/(n+2)$, obtained in [23], where normalization of charge \tilde{g} differs from our, $\tilde{g} = (n+8)g/2$. As discussed above, this result does not correspond to the true zero-dimensional limit of φ^4 theory and its use in the interpolation scheme for improving dependence of g^* on the space dimension d [23] is not reasonable.

saddle points are comparable in value. Then for $K_M(t)$ we have representation

$$K_M(t) = Ae^{i\psi} + A_1e^{i\psi_1} = Ae^{i\psi} (1 + ae^{i\Delta}) \quad (29)$$

and the integral can be turned to zero by the corresponding choice of a and Δ . Indeed, two available degrees of freedom ($\text{Re } t$ and $\text{Im } t$) are in principle sufficient to adjust a and Δ . With variation of t , parameter a surely passes through the unit value, since the complex t plane contains regions where dominates either the first, or the second term of (29). As for the change of Δ , it occurs in infinite limits (see below), and the integral $K_M(t)$ has an infinite number of zeroes lying close to the lines $\chi = \pm 3\pi/4$ and accumulating at infinity. Therefore, the saddle-point approximation used in above considerations can be justified for zeroes lying in the large $|t|$ region.

It is easy to see that the limit $g \rightarrow \infty$ can be achieved, if K_2 goes to zero; then (20,23) are simplified,

$$g \approx -\frac{n}{n+2} \frac{K_4 K_0}{K_2^2}, \quad \beta(g) \approx -\frac{4n}{n+2} \frac{K_4 K_0}{K_2^2}, \quad (30)$$

and the parametric representation can be resolved in the form

$$\beta(g) = 4g, \quad g \rightarrow \infty, \quad (31)$$

while from (26) we have

$$\eta_2(g) = -4, \quad g \rightarrow \infty. \quad (32)$$

In accordance with expectations, the asymptotics of $\beta(g)$ appears to be linear.

In derivation of (31), (32) we did not use the explicit form of the integrals $K_M(t)$: it was essential only that (a) the integral $K_2(t)$ has any zeroes, and (b) zeroes of different integrals $K_M(t)$ do not coincide. Let us show that it is indeed so. The values of action for the saddle points $\varphi = 0$ and $\varphi^2 = -t/2$ are equal to 0 and $t^2/4$ correspondingly, and contributions of these points are comparable for $\text{Re } t^2 = 0$ or $\chi = \pm\pi/4, \pm 3\pi/4$. However, values $\chi = \pm\pi/4$ are not suitable: the integral $K_M(t)$ exhibits the Stokes phenomenon, which is related with the change of topology for lines of the steepest descent (see, e.g. [22]). This change of topology occurs at $\chi = \pm\pi/2$: for $0 < |\chi| < \pi/2$ the line of the steepest descent passes only the trivial saddle point (Fig. 2,a), while for $\pi/2 < |\chi| < \pi$ both saddle points are passed (Fig. 2,b). The compensation of two contributions (29) is possible for $\chi = \pm 3\pi/4$, but does not occur for $\chi = \pm\pi/4$. Setting $t = \rho e^{i\chi}$, $\rho \gg 1$, $\chi = 3\pi/4 + \delta$, $\delta \ll 1$, we have for contributions of two saddle points in the integral $K_0(t)$

$$K_0(t) = \rho^{-n/2} e^{-i\frac{3\pi}{8}n} \left[\frac{1}{2} \Gamma\left(\frac{n}{2}\right) + \frac{\sqrt{\pi}}{2^{n/2}} e^{-i\frac{\pi}{4} + i\frac{\pi}{4}n - i\frac{1}{4}\rho^2} \rho^{n-1} e^{\frac{1}{2}\rho^2\delta} \right] \quad (33)$$

Choosing $\delta(\rho)$ from the condition

$$\rho^{n-1} e^{\frac{1}{2}\rho^2\delta} = \frac{2^{n/2-1}}{\sqrt{\pi}} \Gamma\left(\frac{n}{2}\right), \quad \text{i.e.} \quad \delta \sim \ln \rho / \rho^2 \quad (34)$$

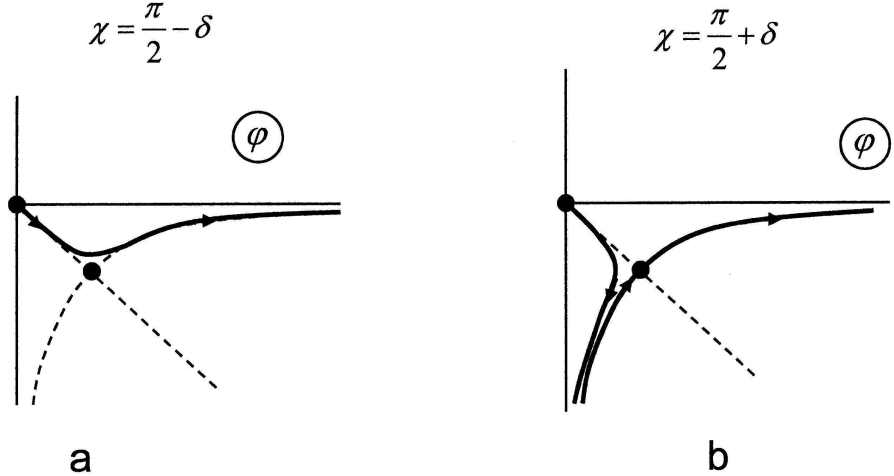


Figure 2: The lines of the steepest descent for the integral $K_M(t)$: (a) for $0 < \chi < \pi/2$, and (b) for $\pi/2 < \chi < \pi$.

we obtain

$$K_0(t) = \frac{1}{2} \Gamma\left(\frac{n}{2}\right) \rho^{-n/2} e^{-i\frac{3\pi}{8}n} \left[1 + e^{\frac{i}{4}(\pi + \pi n - \rho^2)}\right] \quad (35)$$

and zeroes of $K_0(t)$ lie in the points

$$\rho_s^2 = \pi(n + 5) + 8\pi s, \quad s\text{-integer}. \quad (36)$$

The results for $K_M(t)$ can be obtained by the change $n \rightarrow n + M$, and it is clear from (34), (36) that different integrals $K_M(t)$ turn to zero in different points.

4. General d -dimensional case

According to (24), the complex t values with $|t| \rightarrow \infty$ correspond to complex g_0 with $|g_0| \rightarrow 0$, and we come to miraculous conclusion: large values of the renormalized charge g corresponds not to large values of the bare charge g_0 (as naturally to think⁵), but to its

⁵It is commonly accepted that some universal function $g = f(L)$ can be introduced, describing the dependence of the charge on the length scale. Then the observable charge corresponds to $g_{obs} = f(m^{-1})$, the bare charge corresponds to $g_0 = f(\Lambda^{-1})$, and the renormalized charge defined at the scale L , is simply $g = f(L)$, i.e. all charges entering the theory are in fact one and the same charge but related with different scales. However, it is well-known that this picture is approximate due to ambiguity of the renormalization scheme. Definitions of the bare and renormalized charge are technically different and introduced in the cut-off and subtraction schemes, correspondingly [24]. Associated functions $g_0 = f_1(L)$ and $g = f_2(L)$ coincide on the one-loop and two-loop level, but differ in higher orders. Hence, our intuition is relevant only in the weak coupling region.

complex values; more than that, it is sufficient to consider the region $|g_0| \ll 1$, where the saddle-point approximation is applicable.

As a result, the zeroes of the functional integrals can be obtained by compensation of the saddle-point contributions of trivial vacuum and of the instanton configuration with the minimal action⁶. The latter contribution is well-studied in relation with calculation of the Lipatov asymptotics and for $d < 4$ is given by expression (see, e.g. [26], Eq.93)

$$\left[Z_{\alpha_1 \dots \alpha_M}^{(M)}(p_1, \dots, p_M) \right]^{inst} = ic_M (-g_0)^{-(M+r)/2} e^{-S_0/g_0} \langle \phi_c \rangle_{p_1} \dots \langle \phi_c \rangle_{p_M} I_{\alpha_1 \dots \alpha_M} \quad (37)$$

and by somewhat more complicated expression for $d = 4$. Here $\langle \phi_c \rangle_p$ is the Fourier transform of the dimensionless instanton configuration $\phi_c(x)$, S_0 is the corresponding action, r is the number of zero modes, and c_M is a certain constant. Then for $M = 0, 2, \dots$ we have

$$Z_0 = 1 + ic_0 (-g_0)^{-r/2} e^{-S_0/g_0},$$

$$Z_{\alpha\beta}^{(2)}(p, p') = \frac{\delta_{\alpha\beta}}{p^2 + m_0^2} + ic_2 (-g_0)^{-(r+2)/2} e^{-S_0/g_0} \langle \phi_c \rangle_p^2 \delta_{\alpha\beta}, \quad (38)$$

etc., where all contributions are normalized by a value $Z^{(0)}$ at $g = 0$. Setting $t^2 = -S_0/g_0$, we come to expression of type (33), which can be analyzed analogously. It is easy to be convinced that different integrals K_M and their derivatives over m_0^2 have zeroes in different points.

Now we need representation of RG functions in terms of functional integrals. The Fourier transform of (10) is

$$Z_{\alpha_1 \dots \alpha_M}^{(M)}(p_1, \dots, p_M) \mathcal{N} \delta_{p_1 + \dots + p_M} = \sum_{x_1, \dots, x_M} Z_{\alpha_1 \dots \alpha_M}^{(M)}(x_1, \dots, x_M) e^{ip_1 x_1 + \dots + ip_M x_M} \quad (39)$$

where \mathcal{N} is the number of sites on the lattice, which is implied in definition of the functional integral. For the choice of external momenta corresponding to the symmetric point, $p_i \cdot p_j = p^2(4\delta_{ij} - 1)/3$, it is possible to extract factors $I_{\alpha_1 \dots \alpha_M}$ from $Z^{(M)}$, in analogy with (13)

$$Z^{(0)} = K_0, \quad Z_{\alpha\beta}^{(2)}(p, -p) = K_2(p) \delta_{\alpha\beta}, \quad Z_{\alpha\beta\gamma\delta}^{(4)}\{p_i\} = K_4\{p_i\} I_{\alpha\beta\gamma\delta} \quad (40)$$

Introducing vertex $\Gamma^{(0,4)}$ by relation

$$G_{\alpha\beta\gamma\delta}^{(4)}(p_1, \dots, p_4) = G_{\alpha\beta}^{(2)}(p_1) G_{\gamma\delta}^{(2)}(p_3) \mathcal{N} \delta_{p_1+p_2} + G_{\alpha\gamma}^{(2)}(p_1) G_{\beta\delta}^{(2)}(p_2) \mathcal{N} \delta_{p_1+p_3} +$$

$$+ G_{\alpha\delta}^{(2)}(p_1) G_{\beta\gamma}^{(2)}(p_3) \mathcal{N} \delta_{p_1+p_4} - G_{\alpha\alpha'}^{(2)}(p_1) G_{\beta\beta'}^{(2)}(p_2) G_{\gamma\gamma'}^{(2)}(p_3) G_{\delta\delta'}^{(2)}(p_4) \Gamma_{\alpha'\beta'\gamma'\delta'}^{(0,4)}(p_1, \dots, p_4) \quad (41)$$

and extracting $I_{\alpha_1 \dots \alpha_M}$, we have

$$G_{\alpha\beta}^{(2)}(p, -p) = G_2(p) \delta_{\alpha\beta}, \quad G_{\alpha\beta\gamma\delta}^{(4)}\{p_i\} = G_4\{p_i\} I_{\alpha\beta\gamma\delta}, \quad \Gamma_{\alpha\beta\gamma\delta}^{(0,4)}\{p_i\} = \Gamma_4\{p_i\} I_{\alpha\beta\gamma\delta} \quad (42)$$

⁶ Contributions of higher instantons contain additional smallness for $|g_0| \ll 1$.

For strictly zero momenta p_i , the relation of G_4 to Γ_4 contains factors \mathcal{N} , proportional to volume of the system. It is more convenient to set $p_i \sim \mu$, excluding special equalities like $p_1 + p_2 = 0$, and choose μ so that $\mathcal{L}^{-1} \lesssim \mu \ll m$, where lower bound goes to zero in the limit of the infinite system size \mathcal{L} . Then

$$G_4 = \frac{K_4}{K_0}, \quad \Gamma_4 = -\frac{G_4}{G_2^4} = -\frac{K_4 K_0^3}{K_2^4}, \quad (43)$$

where the integrals are taken at zero momenta, and

$$G_2 = \frac{K_2(p)}{K_0}, \quad \Gamma_2(p) = \frac{1}{G_2(p)} = \frac{K_0}{K_2(p)} \approx \frac{K_0}{K_2} + \frac{K_0 \tilde{K}_2}{K_2^2} p^2 \quad (44)$$

where we have written for small p ⁷

$$K_2(p) = K_2 - \tilde{K}_2 p^2 + \dots \quad (45)$$

Expressions for the Z -factors, renormalized mass and charge follows from (7)

$$Z = \left[\frac{\partial}{\partial p^2} \Gamma_2(p) \right]_{p=0}^{-1} = \frac{K_2^2}{K_0 \tilde{K}_2}, \quad (46)$$

$$m^2 = Z \Gamma_2(0) = \frac{K_2}{\tilde{K}_2}, \quad (47)$$

$$g = m^{-\epsilon} Z^2 \Gamma_4 = - \left(\frac{K_2}{\tilde{K}_2} \right)^{d/2} \frac{K_4 K_0}{K_2^2}, \quad (48)$$

$$\frac{1}{Z_2} = \Gamma_{12}\{p_i = 0\} = \frac{d\Gamma_2(0)}{dm_0^2} = \left(\frac{K_0}{K_2} \right)' = \frac{K_0' K_2 - K_0 K_2'}{K_2^2},$$

and

$$\frac{dm^2}{dm_0^2} = \left(\frac{K_2}{\tilde{K}_2} \right)' = \frac{K_2' \tilde{K}_2 - K_2 \tilde{K}_2'}{\tilde{K}_2^2}, \quad (49)$$

where the prime denotes derivatives over m_0^2 . As in Sec.3, parameters g_0 and Λ are considered to be fixed; then m^2 is a function of only m_0^2 and derivative dm^2/dm_0^2 is determined by the expression, inverse to (49). Using definition (9) for RG functions, we have

$$\beta(g) = \left(\frac{K_2}{\tilde{K}_2} \right)^{d/2} \left\{ -d \frac{K_4 K_0}{K_2^2} + 2 \frac{(K_4' K_0 + K_4 K_0') K_2 - 2 K_4 K_0 K_2'}{K_2^2} \frac{\tilde{K}_2}{K_2 \tilde{K}_2' - K_2' \tilde{K}_2} \right\} \quad (50)$$

⁷ The singular p -dependence $\Gamma_2 \sim p^{2-\eta}$ arises near the phase transitions points where $m^2 = 0$. For small m^2 , when the correlation radius ξ is finite, p -dependence remains singular for $p \gg \xi^{-1}$, but it is regular for $p \ll \xi^{-1}$. In the present case, m^2 is finite and in fact tends to infinity in the strong coupling limit; so the p -dependence is surely regular for small p .

$$\eta(g) = -\frac{2K_2\tilde{K}_2}{K_2\tilde{K}'_2 - K'_2\tilde{K}_2} \left[2\frac{K'_2}{K_2} - \frac{K'_0}{K_0} - \frac{\tilde{K}'_2}{\tilde{K}_2} \right] \quad (51)$$

$$\eta_2(g) = \frac{2K_2\tilde{K}_2}{K_2\tilde{K}'_2 - K'_2\tilde{K}_2} \left\{ \frac{K''_0K_2 - K_0K''_2}{K'_0K_2 - K_0K'_2} - 2\frac{K'_2}{K_2} \right\} \quad (52)$$

Eqs.(48), (50), (51), (52) determine $\beta(g)$, $\eta(g)$, $\eta_2(g)$ in the parametric form: for fixed g_0 and Λ , the right hand sides of these equations are the functions of only m_0^2 , while dependence on the specific choice of g_0 and Λ is absent due to general theorems (Sec.2).

If we make the change of variables $\varphi \rightarrow \varphi(u/8)^{-1/4}$, then all integrals are convergent for any value of $t \sim g_0^{-1/2}$ (see (24)) and hence they are regular in the whole complex t plane except of $t = \infty$. Any infinities in the right hand sides of Eqs. 48, 50–52 can be related only with the zeroes of functional integrals.⁸ It is clear from Eq.(48) that the limit $g \rightarrow \infty$ can be achieved by two ways: tending to zero either K_2 , or \tilde{K}_2 . For $\tilde{K}_2 \rightarrow 0$, equations (48) and (50–52) give

$$g = -\left(\frac{K_2}{\tilde{K}_2}\right)^{d/2} \frac{K_4K_0}{K_2^2}, \quad \beta(g) = -d\left(\frac{K_2}{\tilde{K}_2}\right)^{d/2} \frac{K_4K_0}{K_2^2}, \quad \eta(g) \rightarrow 2, \quad \eta_2(g) \rightarrow 0 \quad (53)$$

and the parametric representation is resolved in the form

$$\beta(g) = dg, \quad \eta(g) = 2, \quad \eta_2(g) = 0 \quad (g \rightarrow \infty). \quad (54)$$

For $K_2 \rightarrow 0$, the limit $g \rightarrow \infty$ can be achieved only for $d < 4$:

$$\beta(g) = (d-4)g, \quad \eta(g) = 4, \quad \eta_2(g) \rightarrow 4 \quad (g \rightarrow \infty). \quad (55)$$

The results (54), (55) correspond probably to the different branches of the function $\beta(g)$. It is easy to understand that the physical branch is the first of them. Indeed, it is commonly accepted in phase transitions theory that properties of φ^4 theory change smoothly as a function of space dimension, and results for $d = 2, 3$ can be obtained by analytic continuation from $d = 4 - \epsilon$. All available information indicates on positiveness of $\beta(g)$ for $d = 4$ (Sec.1), and consequently its asymptotics at $g \rightarrow \infty$ is positive; the same property is expected for $d < 4$ by continuity. The result (54) does obey such property, while the branch (55) does not exist for $d = 4$ at all. Eq.54 agrees with the approximate results mentioned in Sec.1 and with the exact result $\beta(g) = 2g$ for the asymptotics of β -function of the 2D Ising model [25], obtained from the duality relation⁹. For $d = 0$, Eq.54 does not agree with (31) by the reasons discussed in Sec.3.

⁸ It is well-known from the phase transitions theory, that singularities of functional integrals can be related only with the points $m^2 = 0$. In this case, the correlation radius ξ is infinite and we really need to have an infinite system and make the singular thermodynamic limit. If $m^2 \neq 0$, then ξ is finite and we can take the system size \mathcal{L} much larger than ξ but finite. If the condition $\mathcal{L} \gg \xi \gg \Lambda^{-1}$ is fulfilled, the functional integrals can be approximated by finite-dimensional ones and have no singularities for finite t .

⁹ Definition of the β -function in [25] differs by the sign from the present paper.

5. Concluding remarks

According to above considerations, the standard renormalization procedure defines theory for $0 \leq g \leq g_{max}$, where g_{max} is finite. For values $g_{max} < g < \infty$, the theory is defined by analytic continuation, and large values of g correspond to complex values of g_0 . The latter situation looks inadmissible: the S -matrix can be expressed through the Dyson T -exponential of the bare action, and Hermiticity of the bare Hamiltonian looks crucial for unitarity of theory.

In fact, a situation is more complicated, as demonstrated by Bogolyubov's axiomatical construction of the S -matrix [2]: according to it, the general form of the S -matrix is given by the T -exponential of iA , where A is a sum of (i) the bare action, and (ii) a sequence of arbitrary "integration constants" which are determined by quasi-local operators. In the regularized theory we can set the "integration constants" to be zero, and the S -matrix is determined by the bare action. However, in the course of renormalization these constants are taken non-zero, in order to remove divergences (in fact, Bogolyubov's theorem on renormalizability is based on this construction). These non-zero "integration constants" can be absorbed by the action due to the change of its parameters. As a result, for the true continual theory the S -matrix is determined by the renormalized action. Physically, it is quite reasonable because the bare Hamiltonian does not exist and the Schrödinger equation is ill-defined. From this point of view there is no problem with the complex bare parameters, since the renormalized Lagrangian is Hermitian for real g .¹⁰

Some problems remain for regularized theory, where the bare and renormalized Lagrangians are equally admissible and a situation looks controversial: the renormalized Lagrangian is Hermitian and corresponds to unitary theory, while the bare Lagrangian is non-Hermitian and unitarity looks spoiled. The analogous situation was discussed for the exactly solvable Lee model [27], which also has the complex bare coupling for the sufficiently large renormalized coupling. After the paper [28] it was generally accepted and fixed by textbooks [29, 30] that the Lee model is physically unsatisfactory due to existence of "ghost" states (i.e. the states with a negative norm). Quite recently [31] it was found that this point of view is incorrect and the Lee model is completely acceptable physical theory. It is a key idea of [31] (see also [32]) that analytical continuation of the Hamiltonian parameters to the complex plane should be assisted by modification of the inner product for the corresponding Hilbert space. Instead of the usual definition

$$(f, g) = \int f^*(x)g(x)dx$$

the inner product is defined as

$$(f, g)_G = (f, \hat{G}g) = \int f^*(x)G(x, y)g(y)dxdy$$

¹⁰ The bare Hamiltonian should be taken Hermitian in the process of renormalization, since perturbation theory is different for non-Hermitian operators. When the relation of renormalized parameters with bare ones is obtained, it can be analytically continued; possibility of such continuation is clear from Eqs.47,48.

and with the proper choice of the operator \hat{G} the bare Hamiltonian is Hermitian in respect to this inner product. As a result, all states of the Lee model have a positive norm and evolution is unitary. Modification of the inner product does not imply any revision of quantum mechanical axioms, if it is applied to formally defined bare Hamiltonians related with artificial constructions (like an auxiliary lattice) and not existing in reality. The interesting question arises, is such modification admissible in physically relevant situations¹¹ [32]; in fact, arising of non-locality (the kernel $G(x, y)$ instead $\delta(x - y)$) is rather natural consequence of regularization.

The analogous procedure should exist in the present case, in order to remove controversy and give possibility to define the β -function and anomalous dimensions for all positive g . This procedure can be put in a more general context. The arbitrary choice of non-zero "integration constants" in Bogolyubov's construction allows to express the S -matrix through different renormalized Lagrangians: it is a well-known ambiguity of the renormalization scheme [24] corresponding to different definitions of renormalized parameters. Replacement of one definition of charge (g) for another (\tilde{g}) corresponds to a change of variables $g = f(\tilde{g})$, not affecting the values of observable quantities but giving a different parametrization for them. Let theory is satisfactory for some specific definition of g ; then there exist a lot of other satisfactory definitions given by "good" functions $g = f(\tilde{g})$. Also, there are a lot of definitions given by "bad" functions (e.g. singular or complex), for which the theory is looking not satisfactory. From this point of view, the paper [31] gives a constructive interpretation for a "bad" change of variables $g = f(g_0)$ (the bare charge in regularized theory can be consider as a particular definition of the renormalized charge).

Our results for the β -function confirm that accepted definition of the renormalized parameters (Eqs. 47, 48) is satisfactory. Indeed, one can accept the arbitrary positive value for renormalized charge g at the fixed scale μ , while its values for other scales are determined by the Gell-Mann – Low equation. This definition does not possess any pathologies like the Landau pole.

Few comments should be given for the dependence of our result (54) on the renormalization scheme. From the mathematical side, the change of variables $g = f(\tilde{g})$ is completely arbitrary. Physically, we should accept some restrictions on it, if we want that both definitions g and \tilde{g} were definitions of "charge", i.e. give some measure of the vertex Γ_4 . As the minimal physical restriction we can accept that the function $g = f(\tilde{g})$ should be regular and give one-to-one correspondence for physical values of g and \tilde{g} . If such restriction is accepted, then the change of variables $g = f(\tilde{g})$ does not allow to transfer between three qualitatively different situations in the Bogolyubov and Shirkov classification (see [33] for details). If the "zero charge" situation is absent for one reasonable definition of charge, it

¹¹ If the answer is negative, then large values of renormalized coupling are inaccessible in the condensed matter applications of φ^4 theory (where the lattice bare Hamiltonians have a physical sense). However, analytical continuation of the β -function to arbitrary g is useful from viewpoint of summation of perturbation series. It should be clear from [3, 6] that knowledge of the strong coupling asymptotics essentially simplifies the summation procedure and makes it well-defined and more efficient. On the other hand, the value g_{max} depends on the specific lattice and can be large in some cases.

will be absent for another.

Less rigorously, we can argue that two "physical" definitions of charge differ in the manner, by which the vertex Γ_4 is related with a length scale L . If dependence on L has a power-law character, then the order of magnitude uncertainty in L gives a factor of the order of unity in definition of charge; so $g \sim \tilde{g}$. Then the linear asymptotics $\beta(g) = \beta_\infty g$ in one case produces the same asymptotics in the other case, as can be proved by contradiction (since $g \sim L^{-\beta_\infty}$ for small L , then the difference in β_∞ violates the relation $g \sim \tilde{g}$).

In conclusion, the strong coupling asymptotics of the β -function in φ^4 theory is shown to be linear in the general d -dimensional case. In four dimensions, it means possibility to construct continuous theory with finite interaction at large distances.

The author is indebted for discussions to the participants of L.N.Lipatov's seminar in Petersburg Nuclear Physics Institute and of B.L.Ioffe's seminar in Institute of Theoretical and Experimental Physics; he is also grateful to C.M.Bender for discussion of the Lee model.

References

- [1] L. D. Landau, A. A. Abrikosov, and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR **95**, 497, 773, 1177 (1954).
- [2] N. N. Bogolyubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields, 3rd ed. (Nauka, Moscow, 1976; Wiley, New York, 1980).
- [3] I. M. Suslov, Zh. Eksp. Teor. Fiz. **120**, 5 (2001) [JETP **93**, 1 (2001)].
- [4] I. M. Suslov, Pis'ma Zh. Eksp. Teor. Fiz. **74**, 211 (2001) [JETP Lett. **74**, 191 (2001)].
- [5] I. M. Suslov, Pis'ma Zh. Eksp. Teor. Fiz. **76**, 387 (2002) [JETP Lett. **76**, 327 (2002)].
- [6] I. M. Suslov, Zh. Eksp. Teor. Fiz. **127**, 1350 (2005) [JETP **100**, 1188 (2005)].
- [7] L. N. Lipatov, Zh. Eksp. Teor. Fiz. **72**, 411 (1977) [Sov.Phys. JETP **45**, 216 (1977)].
- [8] 't Hooft G, in: The whys of subnuclear physics (Erice, 1977), ed. A Zichichi, Plenum Press, New York, 1979.
- [9] Beneke M, Phys. Rept. **317**, 1 (1999), Sec. 2.4.
- [10] I. M. Suslov, Zh. Eksp. Teor. Fiz. **126**, 542 (2004) [JETP **99**, 474 (2004)].
- [11] I. M. Suslov, Zh. Eksp. Teor. Fiz. **116**, 369 (1999) [JETP **89**, 197 (1999)].

- [12] A. A. Pogorelov, I. M. Suslov, Zh. Eksp. Teor. Fiz. **132**, 406 (2007) [JETP **105**, 360 (2007)].
- [13] A. A. Pogorelov, I. M. Suslov, Pis'ma Zh. Eksp. Teor. Fiz. **86**, 41 (2007) [JETP Lett. **86**, 39 (2007)].
- [14] P. Butera, M. Comi, hep-th/0112225.
- [15] K. G. Wilson and J. Kogut, Phys. Rep. C **12**, 75 (1975).
- [16] J. Frölich, Nucl. Phys. B **200** [FS4], 281 (1982); M. Aizenman, Commun. Math. Soc. **86**, 1 (1982).
- [17] I. M. Suslov, arXiv: 0804.2650
- [18] I. M. Suslov, arXiv: 0806.0789
- [19] E. Brezin, J. C. Le Guillou, J. Zinn-Justin, in Phase Transitions and Critical Phenomena, ed. by C. Domb and M. S. Green, Academic, New York (1976), Vol. VI.
- [20] G. A. Baker, Jr., B. G. Nickel, D. I. Meiron, Phys. Rev. Lett. **36**, 1351 (1976); Phys. Rev. B **17**, 1365 (1978); J. C. Le Guillou, J. Zinn-Justin, Phys. Rev. Lett. **39**, 95 (1977); Phys. Rev. B **21**, 3976 (1980).
- [21] E. Brezin, J. C. Le Guillou, J. Zinn-Justin, Phys. Rev. D **15**, 1544 (1977).
- [22] A. I. Nikishov, V. I. Ritus, Teor. Mat. Fiz. 92, 24 (1992) [Theor. Math. Phys. **92**, 711 (1992)]; Zh. Eksp. Teor. Fiz. **105**, 769 (1994) [JETP **78**, 411 (1994)].
- [23] A. Pelissetto, E. Vicari, Nucl. Phys. B **519**, 626 (1998).
- [24] A. A. Vladimirov and D. V. Shirkov, Usp. Fiz. Nauk **129**, 407 (1979) [Sov. Phys. Usp. **22**, 860 (1979)].
- [25] G. Jug, B. N. Shalaev, J. Phys. A **32**, 7249 (1999).
- [26] D. A. Lobaskin and I. M. Suslov, Zh. Eksp. Teor. Fiz. 126, 268 (2004) [JETP **99**, 234 (2004)].
- [27] T. D. Lee, Phys. Rev. **95**, 1329 (1954).
- [28] G. Källén, W. Pauli, Mat.-Fyz. Medd. **30**, No.7 (1955).
- [29] S. S. Schweber, An Introduction to Relativistic Quantum Field Theory (Row, Peterson and Co., Evanston, 1961), Ch.12.
- [30] G. Barton, Introduction to Advanced Field Theory (John Willey & Sons, New York, 1963), Ch.12.

- [31] C. M. Bender, S. F. Brandt, J.-H. Chen, Q. Wang, Phys. Rev. D **71**, 025014 (2005).
- [32] C. M. Bender, D. C. Brody, H. F. Jones, Phys. Rev. Lett. **89**, 270401 (2002)
C. M. Bender, Rept. Prog. Phys. **70**, 947 (2007).
- [33] I. M. Suslov, arXiv: hep-ph/0605115.