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ORDER, DISORDER, AND PHASE TRANSITION IN CONDENSED SYSTEM

Renormalization Group Functions of the φ^4 Theory from High-Temperature Expansions

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Abstract—It has been previously shown that calculation of the renormalization group (RG) functions of scalar φ^4 theory reduces to analysis of thermodynamic properties of the Ising model. Using high-temperature expansions for the latter, RG functions of the four-dimensional theory can be calculated for arbitrary coupling constant g with an accuracy of 10^{-4} for the Gell-Mann–Low function $\beta(g)$ and with an accuracy of 10^{-3}–10^{-2} for anomalous dimensions. The expansions of the renormalization group functions up to the 13th order in $g^{-1/2}$ have been obtained.

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1. INTRODUCTION

As was recently shown in [1, 2], the Gell-Mann–Low function $\beta(g)$ and anomalous dimensions of the φ^4 theory can be expressed in terms of the functional integrals, providing the representation

$$g = F(g_0, m_0, \Lambda), \quad \beta(g) = F_1(g_0, m_0, \Lambda), \qquad (1)$$

where g_0 and m_0 are the bare charge and mass, respectively; Λ is the momentum cutoff parameter, and g is the renormalized charge. Large g values are reached only near a zero of one of the functional integrals, where the right-hand sides of Eqs. (1) are significantly simplified and the parametric representation is resolved in the explicit form. As a result, asymptotic expressions for the β function and anomalous dimensions are obtained. A similar approach can also be implemented in QED [3].

Parametric representation (1) has the following general property. If g_0 is expressed in terms of g using the first of Eqs. (1) and the resulting expression is substituted into the second equation, the dependence on m_0 and Λ disappears according to the general theorems [4], so that the β function depends only on g. However, this property is not automatically satisfied in applied calculations. The reason is that the general theorems imply the continual limit $\Lambda \longrightarrow \infty$, which physically means the condition

$$m \ll \Lambda \text{ or } \xi \gg a, \qquad (2)$$

where m is the renormalized mass, ξ is the correlation radius, and $a = \Lambda^{-1}$ is the lattice constant at which the functional integral is defined. Under condition (2) in the region of large g_0 values, the functional integrals of the φ^4 theory are reduced to Ising sums; as a result, Eqs. (1) have the form [2]

$$g = F(\kappa), \quad \beta(g) = F_1(\kappa), \qquad (3)$$

where κ has the meaning of inverse temperature in the Ising model, and it is obvious that the β function depends only on g. Condition (2) formally corresponds to the inequality $-g_0^{-1}m_0^2/\Lambda^2 \gtrsim 1$, but reducting to the Ising model is really possible under the weaker condition

$$\begin{gathered} g_0 \gg 1, \quad -g_0^{-1/2}m_0^2/\Lambda^2 \gg 1, \\ -g_0^{-1}m_0^2/\Lambda^2 = \kappa - \text{arbitrary}. \end{gathered} \qquad (4)$$

For this reason, parametric representation (3) remains valid in the region of small κ values, where g values are large and gradient expansions are applicable.

At first glance, the condition $g_0 \gg 1$ corresponds to the strong coupling regime and parametric representation (3) is limited by only this condition. However, there is another view on this situation. Let us enforce conditions (4) by passing to the limit

$$\begin{gathered} g_0 \longrightarrow \infty, \quad -g_0^{-1/2}m_0^2/\Lambda^2 \longrightarrow \infty, \\ -g_0^{-1}m_0^2/\Lambda^2 = \kappa = \text{const}. \end{gathered} \qquad (5)$$

In this case, the transition from Eqs. (1) to Eqs. (3) is valid without any approximations and conserves strict equivalence with the initial φ^4 theory under a certain choice of its bare parameters; the last property ensures the conservation of the form of the Lagrangian under renormalizations. The passage to the limit $g_0 \longrightarrow \infty$ does not mean the same passage for the renormalized charge g; in fact, according to gradient expansions, g varies from infinity to unity when κ varies from zero to about unity. Since parametric representation (3) is exact and specifies the β function in the interval $1 \lesssim g < \infty$, it can be analytically continued and treated as a definition of $\beta(g)$ at arbitrary g values. However, there is a question: Does this definition provide correct results in the weak-coupling region?

An answer to this question can be obtained using high-temperature series [5]. Such series are traditionally constructed for χ_2, μ_2, and χ_4 (see Section 2), which completely specify the right-hand sides of Eqs. (3). High-temperature expansions are formally applicable for small κ values, but their comparatively large length (up to 30 terms in some cases) allows a successful analysis of the vicinity of the phase transition point κ_c and leads to the results consistent with other methods. Consequently, good approximations for the indicated quantities can be obtained throughout the interval $0 \le \kappa \le \kappa_c$. The substitution of such results into the right-hand sides of Eqs. (3) makes it possible to determine the renormalization group functions in the interval $g^* \le g < \infty$, where g^* is the fixed point of the renormalization group. In the four-dimensional case, $g^* = 0$ and the mentioned procedure completely determines the renormalization group functions.

In many works [6–16], the high-temperature series were used to test logarithmic corrections to scaling [17]. Already those works provide the positive answer to the above question: parametric representation (3) gives correct results in the weak-coupling region. Therefore, the main aim of present paper is to construct the renormalization group functions of the four-dimensional φ^4 theory for arbitrary g values. This can be done with an accuracy of 10^{-4} for the β function and with a slightly lower accuracy for anomalous dimensions.

The determination of calculated renormalization group functions implies the use of a lattice regularization different from the usual Pauli–Villars regularization, isotropic cutoff in the momentum space, dimensional regularization, etc. However, the β function in the used scheme is determined in terms of the observed charge and mass [1, 2] and should be independent of the cutoff method. Such a dependence is possible for anomalous dimensions, because they are determined in terms of the unobservable Z factors. In any case, the distinction of this regularization procedure from the usual procedures is no more than the difference between the latter procedures.

2. INITIAL RELATIONS

Let us consider the n-component φ^4 theory with the action

$$S\{\varphi\} = \int d^d x \left\{ \frac{1}{2}\sum_{\alpha=1}^{n} (\nabla\varphi_\alpha)^2 + \frac{1}{2}m_0^2 \sum_{\alpha=1}^{n}\varphi_\alpha^2 + \frac{1}{8}u_0\left(\sum_{\alpha=1}^{n}\varphi_\alpha^2\right)^2 \right\}, \quad u_0 = g_0\Lambda^\epsilon, \quad \epsilon = 4-d, \tag{6}$$

where g_0 and m_0 are the bare charge and mass, respectively; d is the dimensionality of space; and Λ is the momentum cutoff parameter. The most general functional integral of this theory contains M multipliers of the field φ in the pre-exponential factor,

$$Z^{(M)}_{\alpha_1\ldots\alpha_M}(x_1, \ldots, x_M) = \int D\varphi\varphi_{\alpha_1}(x_1)\varphi_{\alpha_2}(x_2) \ldots \varphi_{\alpha_M}(x_M)\exp(-S\{\varphi\}), \tag{7}$$

and will be denoted as $K_M\{p_i\}$ after the transition to the momentum representation and the separation of δ factors,

$$Z^{(M)}_{\alpha_1\ldots\alpha_M}(p_1, \ldots, p_M) = K_M\{p_i\}\mathcal{N}\delta_{p_1+\ldots+p_M}I_{\alpha_1\ldots\alpha_M}, \tag{8}$$

where $I_{\alpha_1\ldots\alpha_M}$ is the sum of the $\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}\ldots$ with all possible pairings and $\mathcal{N}$ is the number of sites of the lattice on which the functional integral is defined. The integrals $K_M\{p_i\}$ are usually estimated at zero momenta and only one integral $K_2(p)$ is required for small p values,

$$K_2(p) = K_2 - \tilde{K}_2 p^2 + \ldots \tag{9}$$

Below, the case with $d = 4$ and $n = 1$ is considered, but the general formulas are written for arbitrary d and n values.

The below consideration concerns the renormalization group functions $\beta(g)$, $\eta(g)$, and $\eta_2(g)$ entering into the Callan–Symanzik equation [4]

$$\left[\frac{\partial}{\partial\ln m} + \beta(g)\frac{\partial}{\partial g} + \left(L - \frac{N}{2}\right)\eta(g) - L\eta_2(g)\right] \times \Gamma^{(L,N)} = 0, \tag{10}$$

for the vertex $\Gamma^{(L,N)}$ with N external lines of the field φ and L external interaction lines. The expression of these functions in terms of the functional integrals leads to the parametric representation [2]

$$g = -\left(\frac{K_2}{\tilde{K}_2}\right)^{d/2}\frac{K_4K_0}{K_2^2}, \tag{11}$$

$$\beta(g) = -\left(\frac{K_2}{\tilde{K}_2}\right)^{d/2} \times \frac{K_4K_0}{K_2^2}\left\{d + 2\frac{(\ln K_4K_0/K_2^2)'}{(\ln K_2/\tilde{K}_2)'}\right\}, \tag{12}$$

$$\eta(g) = 2\frac{(\ln K_2/K_0)' + (\ln K_2/\tilde{K}_2)'}{(\ln K_2/\tilde{K}_2)'}, \tag{13}$$

$$\eta_2(g) = -2\frac{(\ln K_0/K_2)'' + [(\ln K_0/K_2)']^2}{(\ln K_2/\tilde{K}_2)'(\ln K_0/K_2)'}, \tag{14}$$

where primes stand for derivatives with respect to m_0^2. Under condition (4), the functional integral of the scalar theory can be written in the form [2]

$$Z_M\{\mathbf{x}_i\} = (2\kappa)^{\frac{\mathcal{N}+M}{2}} \int\left(\prod_{\mathbf{x}} d\varphi_{\mathbf{x}}\right)\varphi_{\mathbf{x}_1}\ldots\varphi_{\mathbf{x}_M} \times \exp\left\{-\kappa\sum_{\mathbf{x},\mathbf{x}'} J_{\mathbf{x}-\mathbf{x}'}\varphi_{\mathbf{x}}\varphi_{\mathbf{x}'}\right\}\prod_{\mathbf{x}}\delta(\varphi_{\mathbf{x}}^2-1) \tag{15}$$

and is transformed to an Ising sum over the values $\varphi_{\mathbf{x}} = \pm 1$.

The quantities studied in high-temperature expansions are introduced as

$$\chi_2 = \sum_{\mathbf{x}}\langle\varphi_{\mathbf{x}}\varphi_0\rangle^{\mathrm{c}}, \quad \mu_2 = \sum_{\mathbf{x}}\mathbf{x}^2\langle\varphi_{\mathbf{x}}\varphi_{\mathbf{0}}\rangle^{\mathrm{c}}, \quad \chi_4 = \sum_{\mathbf{x},\mathbf{y},\mathbf{z}}\langle\varphi_{\mathbf{x}}\varphi_{\mathbf{y}}\varphi_{\mathbf{z}}\varphi_{\mathbf{0}}\rangle^{\mathrm{c}}, \tag{16}$$

where superscript "c" marks the connected diagrams, and coincides up to factors with the ratios K_2/K_0, $\tilde{K}_2/K_0$, and K_4/K_0 of the functional integrals introduced above; more precisely,

$$\frac{K_2}{\tilde{K}_2} = 2d\frac{\chi_2}{\mu_2} \equiv \frac{1}{\kappa}f_0(\kappa), \quad \frac{K_2}{K_0} = 2\kappa\chi_2 \equiv \kappa f_2(\kappa), \quad \frac{K_4K_0}{K_2^2} = \frac{1}{3}\frac{\chi_4}{\chi_2^2} \equiv -f_4(\kappa), \tag{17}$$

where the introduced functions $f_0(\kappa)$, $f_2(\kappa)$, and $f_4(\kappa)$ will be used below. It was taken into account that there is no zeroth term in the expansion of μ_2 in κ (see Eq. (20) below), so that all functions $f_0(\kappa)$, $f_2(\kappa)$, and $f_4(\kappa)$ are regular and their expansions begin with the zeroth term. The substitution of Eqs. (17) into Eqs. (11)–(14) gives

$$g = \left(\frac{f_0(\kappa)}{\kappa}\right)^{d/2} f_4(\kappa), \quad \frac{\beta(g)}{g} = d - 2\kappa\frac{[\ln f_4(\kappa)]'}{1-\kappa[\ln f_0(\kappa)]'}, \quad \eta(g) = -2\kappa\frac{[\ln f_0(\kappa)f_2(\kappa)]'}{1-\kappa[\ln f_0(\kappa)]'}, \tag{18}$$

$$\eta_2(g) = -2\frac{(1+\kappa[\ln f_2(\kappa)']^2 + 1 - \kappa^2[\ln f_2(\kappa)]''}{(1-\kappa[\ln f_0(\kappa)]')(1+\kappa[\ln f_2(\kappa)]')}.$$

Taking the limit $\kappa \longrightarrow 0$, it is easy to obtain strong coupling behavior for the RG functions [2]:

$$\beta(g) = dg, \ \eta(g) = 0, \ \eta_2(g) = -4 \ (g \longrightarrow \infty). \tag{19}$$

For a simple hypercubic lattice with the interaction between the nearest neighbors, the first terms of the expansion of functions (16) for $d = 4$ and $n = 1$ have the form [18]

$$\chi_2 = 1 + 16\kappa + 224\kappa^2 + \ldots, \quad \mu_2 = 16\kappa + 512\kappa^2 + 33920/3\kappa^3 + \ldots, \quad \chi_4 = -2 - 128\kappa - 4672\kappa^2 - \ldots \tag{20}$$

The substitution into Eqs. (18) makes it possible to obtain the expansion of the renormalization group functions in $g^{-2/d}$, in particular, a more accurate asymptotic expression for $\eta(g)$:

$$\eta(g) = \frac{16}{9}\frac{1}{g}, \quad g \longrightarrow \infty. \tag{21}$$

The universality of this asymptotics has not been tested and, strictly speaking, it refers to the indicated model. Below, 14 terms of expansion (20) presented for $n = 1$ in tables 5, 8, and 11 in [18] are used.

3. VICINITY OF THE PHASE TRANSITION

3.1. General Strategy

The foundation of the application of high-temperature series for investigating the critical behavior is as follows. Let a certain quantity $F(\kappa)$ have a power-law behavior near the transition point $k_{\mathrm{c}} = 1/T_{\mathrm{c}}$,

$$F \propto (T - T_{\mathrm{c}})^{-\lambda} \propto (\kappa_{\mathrm{c}} - \kappa)^{-\lambda}. \tag{22}$$

In this case, the convergence radius of the expansion in κ is limited by the quantity κ_{c}. In actual cases, κ_{c} is the nearest singularity to the coordinate origin; this circumstance facilitates its analysis. It is easily seen that the nearest singularity for the logarithmic derivative

$$(\ln F)' = \frac{F'}{F} \sim \frac{-\lambda}{\kappa - \kappa_{\mathrm{c}}} \tag{23}$$

is a simple pole with a residue of $-\lambda$ and can be investigated using the Padé approximation. The Padé approximant $[M/N]$ is defined as the ratio of the polynomials of the degrees M and N,

$$(\ln F)' = \frac{P_M(\kappa)}{Q_N(\kappa)} = \frac{p_0 + p_1\kappa + \ldots + p_M\kappa^M}{1 + q_1\kappa + \ldots + q_N\kappa^N}, \tag{24}$$

whose coefficients are chosen such that the first $M + N + 1$ coefficients of the expansion of $(\ln F)'$ in κ are reproduced. It is known that Padé approximants well predict the nearest singularities of the approximated function if these singularities are simple poles [5, 19]. Diagonal ($M = N$) or quasi-diagonal ($M \approx N$) approximants are usually used for which convergence to the

Table 1. Position of the pole corresponding to the critical point κ_c and residue at it (in parentheses) for the indicated functions $[\ln f_i(\kappa)]'$

N	$[\ln f_0(\kappa)]'$	$[\ln f_2(\kappa)]'$	$[\ln f_4(\kappa)]'$
2	0.07519 (1.130)	0.07510 (−1.113)	0.07442 (−1.832)
3	0.07521 (1.131)*	0.07543 (−1.085)	0.07419 (−1.814)
4	0.07502 (1.116)	0.07497 (−1.101)	0.07476 (−1.879)
5	0.07480 (1.063)	0.07513 (−1.103)	0.07477 (−1.881)
6	0.07486 (1.082)	0.07490 (−1.088)	0.07476 (−1.879)

Note: The asterisk in Tables 1–4 marks defect approximants. A "defect" in the Padé analysis is the appearance of a pair of a pole and a root close to each other; as a result, the corresponding Padé approximant is reduced to a lower-order approximant. The defectiveness of the approximant can lead to loose of the accuracy and is a reason for its discrimination.

Table 2. Position of the pole corresponding to the critical point κ_c and residue at it (in parentheses) for the indicated Padé approximants of function $[\ln f_2 f_4]'$

N	$[N+1/N]$	$[N/N]$	$[N/N+1]$
2	0.07418 (−2.871)	0.07461 (−2.936)	0.07558 (−2.963)
3	0.07488 (−2.993)	0.07450 (−2.923)	0.07465 (−2.946)
4	0.07486 (−2.988)	0.07485 (−2.986)	0.07486 (−2.988)
5	0.07487 (−2.989)	0.07486 (−2.987)*	0.07491 (−2.998)*
6	0.07481 (−2.970)	0.07484 (−2.983)	0.07483 (−2.978)

corresponding function is proved under the most general assumptions.

The use of this strategy in the four-dimensional case is complicated by the existence of logarithmic corrections to scaling [4, 17]:

$$\chi_2 \sim \tau^{-1}|\ln\tau|^p, \quad \xi^2 \sim \frac{\mu_2}{\chi_2} \sim \tau^{-1}|\ln\tau|^p,$$
$$\chi_4 \sim \tau^{-4}|\ln\tau|^{4p-1}, \quad p = -\frac{\zeta_1}{\beta_2} = \frac{n+2}{n+8}, \tag{25}$$

where $\tau \sim (\kappa_c - \kappa)$ is the distance to the transition and the exponent p is determined by the first terms of the expansion of the renormalization group functions,

$$\beta(g) = \beta_2 g^2 + \beta_3 g^3 + \ldots,$$
$$\eta(g) = \delta_2 g^2 + \delta_3 g^3 + \ldots, \tag{26}$$
$$\eta_2(g) = \zeta_1 g + \zeta_2 g^2 + \ldots,$$

where

$$\beta_2 = S_4\frac{n+8}{2}, \quad \beta_3 = -S_4^2\frac{9n+42}{4},$$
$$\delta_2 = S_4^2\frac{n+2}{8}, \quad \zeta_1 = -S_4\frac{n+2}{2} \tag{27}$$

and $S_4 = 1/8\pi^2$. According to Eqs. (25),

$$f_0 \sim \tau|\ln\tau|^{-p}, \quad f_2 \sim \tau^{-1}|\ln\tau|^p,$$
$$f_4 \sim \tau^{-2}|\ln\tau|^{2p-1}. \tag{28}$$

The behavior of the charge g is given by the expression

$$g = \frac{c_0}{|\ln\tau|}, \quad c_0 = \frac{2}{\beta_2} \quad (\tau \longrightarrow 0), \tag{29}$$

where the coefficient of the logarithmic factor is universal. When Eqs. (28) and (29) are valid, parametric representation (18) automatically ensures the results $\beta(g) = \beta_2 g^2$, $\eta(g) = 0 \cdot g$, and $\eta_2(g) = \zeta_1 g$, i.e., the correct behavior of the renormalization group functions at small g values.

The objective test of Eqs. (25) for lattice models were performed in many works [6–15]. In particular, it was convincingly shown in [6, 7] that the high-temperature series for the Ising model allow reliable prediction of the exponent p. Expression (29) was confirmed with a satisfactorily accuracy in [7, 9]. Already these results provide the positive answer to the question formulated in the Introduction: parametric representation (18) gives correct results for the renormalization group functions in the weak-coupling region.

3.2. Zeroth Approximation

The Padé analysis of Eqs. (28) is performed by the successive approximation method. The logarithmic factors are ignored in the zeroth approximation and the functions f_i are processed under the assumption of their power-law dependence on τ. The results of such an analysis presented in Table 1 show a significant difference of the exponents from the exact values (see

Table 3. Padé analysis of the functions $\tilde{f}_i(\kappa)$ introduced according to Eqs. (31)

A_0	$[\ln\tilde{f}_0(\kappa)]'$ [6/6], $\bar{\kappa}_c = 0.074842$	$[\ln\tilde{f}_2(\kappa)]'$ [6/6], $\bar{\kappa}_c = 0.074834$	$[\ln\tilde{f}_4(\kappa)]'$ [6/5], $\bar{\kappa}_c = 0.074890$
2.0	0.07491 (1.037)*	0.07486 (−1.023)	0.07488 (−1.968)*
1.0	0.07487 (1.018)	0.074844 (−1.007)	0.07493 (−1.989)*
0.625	–	0.074834 (−1.00005)	–
0.5	0.074855 (1.0085)	0.07482 (−0.996)	0.07477 (−1.960)
0.25	0.074846 (1.0029)	0.07475 (−0.973)	0.07488 (−1.996)
0.2	–	–	0.074890 (−1.9994)
0.13	0.0748420 (1.00005)	–	–
0.1	0.074840 (0.9993)	0.07483 (−0.990)*	0.07490 (−2.0044)
0.06	0.07487 (1.0033)	0.07482 (−0.988)*	0.07491 (−2.0063)

Eqs. (28)) and provide a rough estimate of the critical point

$$\kappa_c = 0.07476–0.07490.$$

A more accurate estimate of κ_c can be obtained taking into account that the ratio $\chi_4/\chi_2 \sim f_4 f_2$ in the scalar case (when $p = 1/3$) behaves as τ^{-3} and contains no logarithms [6]. As is seen in Table 2, the Padé analysis of this quantity really provides the exponent close to the exact value and the corresponding estimate

$$\kappa_c = 0.07481–0.07487, \qquad (30)$$

is almost final and will be only slightly refined below. The central value of interval (30) almost coincides with the result $\kappa_c = 0.074834(15)$ obtained in [6] with a more sophisticated processing.

3.3. First Approximation

In this approximation, the following representation is used:

$$f_0 = \tilde{f}_0|\ln\tau|^{-p}, \quad f_2 = \tilde{f}_2|\ln\tau|^{p}, \quad f_4 = \tilde{f}_4|\ln\tau|^{2p-1} \qquad (31)$$

and the Padé analysis is applied to the functions $\tilde{f}_i$. Since the relation $\tau = A(\kappa_c - \kappa)$ includes the nonuniversal factor A, it can be accepted that

$$|\ln\tau| = A_0 - \ln\left(1 - \frac{\kappa}{\bar{\kappa}_c}\right), \qquad (32)$$

where the free parameter A_0 and trial value $\bar{\kappa}_c$ for the critical point are used to accurately fit the exponent and to obtain a self-consistent result for κ_c. According to Table 3, such a fit is easy and good results for the exponent are obtained in a wide range of the A_0 values. The optimal A_0 values lie in the interval 0.13–0.63 and a new estimate of the critical point,

$$\kappa_c = 0.07483–0.7489,$$

is only slightly shifted as compared to Eq. (30). The results for the constant c_0 in Eq. (29) are shown in Fig. 1a; they are close to the theoretical value $c_0^{\text{th}} = 35.09$, but are systematically above it.

Similar inaccuracies in the determination of c_0 were observed in other works. The use of constants A, B, and D for a simple hypercubic lattice from Table 5 in [7] gives the estimate $c_0 = B/A^2D^4 = 142.8$ instead of

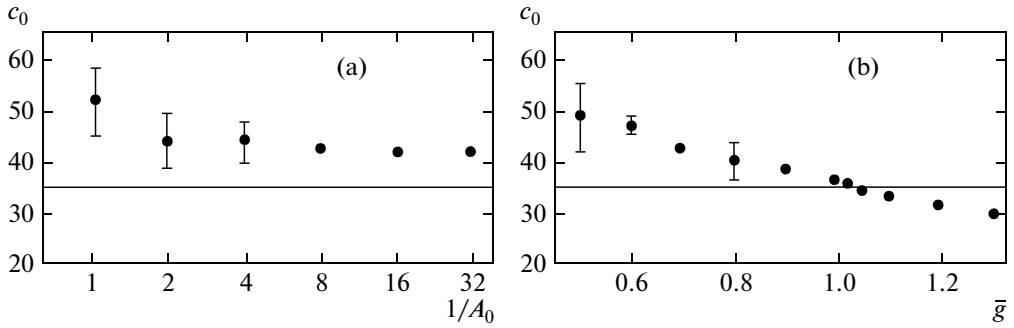

Fig. 1. Constant c_0 in Eq. (29) (a) versus the parameter A_0 in the leading logarithmic approximation and (b) versus $\bar{g}$ in the next-to-leading logarithmic approximation.

Table 4. Padé analysis of the functions $\tilde{f}_i(\kappa)$ introduced according to Eqs. (37)

$\bar{g}$	$[\ln\tilde{f}_0(\kappa)]'$ [6/6], $\bar{\kappa}_c = 0.074843$	$[\ln\tilde{f}_2(\kappa)]'$ [6/6], $\bar{\kappa}_c = 0.074840$	$[\ln\tilde{f}_4(\kappa)]'$ [6/5], $\bar{\kappa}_c = 0.074867$
0.5	0.07492 (1.036)*	0.07488 (−1.024)	0.07487 (−1.968)*
0.7	0.07488 (1.019)*	0.07485 (−1.0096)	0.07491 (−1.988)*
0.85	–	0.074840 (−1.0008)	–
0.9	0.07485 (1.0052)	0.074836 (−0.998)	0.074877 (−1.994)
0.99	0.074843 (1.00005)	–	–
1.0	0.074842 (0.9995)	0.07483 (−0.994)	0.074865 (−1.997)
1.06	–	–	0.074867 (−2.0001)
1.2	0.07482 (0.988)	0.07476 (−0.976)*	0.07488 (−2.010)

a theoretical result of 105.2 referring to the used normalization. A worse estimate was obtained in [10]; very bad results (discrepancies of 9 and 18 times) were obtained for other lattices [7]. A satisfactory test of Eq. (29) was declared in [9], where the tested relation was not Eq. (29), but its consequence $dg^{-1}/d\ln\tau = 1/c_0$; in this case, the central value c_0 approximately corresponds to Fig. 1a and the agreement with the theory was achieved at the expense of an increase in the uncertainty of the results because of differentiation.

3.4. Second Approximation

Expressions (25) and (28) are obtained in the leading logarithmic approximation. In the next-to-leading logarithmic approximation (see Appendix A), they have the form

$$f_0 = h_0\tau(f_{\rm sing})^{-p}, \quad f_2 = h_2\tau^{-1}(f_{\rm sing})^p h_{\rm sing}, \quad f_4 = h_4\tau^{-2}(f_{\rm sing})^{2p-1}. \tag{33}$$

Here, the functions $h_i(\kappa)$ are regular at $\kappa \longrightarrow \kappa_c$ and singular functions are chosen in the form

$$f_{\rm sing}(\kappa) = 1 - \bar{g}\ln\tau + s\bar{g}\ln(1 - \bar{g}\ln\tau), \tag{34}$$

$$h_{\rm sing}(\kappa) = 1 + q\frac{\bar{g}}{f_{\rm sing}(\kappa)}, \quad \tau = 1 - \frac{\kappa}{\kappa_c}, \tag{35}$$

where

$$s = \frac{2\beta_3}{\beta_2^2} - \frac{\zeta_1}{\beta_2} = \frac{n^2 - 8n - 68}{(n+8)^2}, \quad q = \frac{2\delta_2}{\beta_2^2} = \frac{n+2}{(n+8)^2}. \tag{36}$$

The main distinction from Eqs. (28) is reduced to the replacement of $|\ln\tau|$ by $|\ln\tau| + s\ln|\ln\tau|$ with a known parameter s; in view of the ambiguity of the normalization of τ, it is necessary to consider the combinations $A + |\ln\tau| + s\ln(B + |\ln\tau|)$, where the constants A and B are different for different functions. Formally, these constants do not affect the character of a singularity, but their unsuccessful choice can strongly distort the results. To avoid a large number of fitting parameters, $f_{\rm sing}(\kappa)$ was taken in the functional form following from perturbation theory. A reason for such a choice is as follows. The parameter $\bar{g}$ has the meaning of the Ginzburg number and determines the size of the critical region, where logarithmic corrections are significant. It is of interest to estimate this parameter, because the Ginzburg number is often small even in the absence of theoretical reasons for this. The function $f_{\rm sing}(\kappa)$ at small $\bar{g}$ values is close to unity almost everywhere, but increases sharply near κ_c. If the singularity is separated inappropriately, regular functions $h_i(\kappa)$ in Eqs. (33) are rapidly varying near κ_c and are poorly reproduced by Padé approximants. However, the form of Eq. (34) for small $\bar{g}$ values is practically exact, so that the functions $h_i(\kappa)$ are almost constant. The form of Eq. (34) for $\bar{g} \gtrsim 1$ is not exactly correct, but inaccuracy in the separation of singularities in this case is not so critical, because the function $f_{\rm sing}(\kappa)$ is a rather slowly varying function.

The universal choice of $f_{\rm sing}(\kappa)$ for all functions is possible if the $O(\bar{g})$ contributions are negligible as compared to unity (see Appendix A), so that the inclusion of factors of the $h_{\rm sing}(\kappa)$ type is strictly speaking an excess of accuracy. However, such factors are sometimes of qualitative importance and are taken into account in Eqs. (33) in the minimal manner: the product f_0f_2 in this form has the correct singularity and ensures the correct behavior of $\eta(g)$ at small g values; similarly, the product f_4f_2 is incompletely free of logarithms and this property makes it possible to slightly correct deflections observed in Table 2.

Table 4 presents the Padé analysis of the functions $\tilde{f}_i$ introduced by the relations

$$f_0 = \tilde{f}_0(f_{\rm sing})^{-p}, \quad f_2 = \tilde{f}_2(f_{\rm sing})^p h_{\rm sing}, \quad f_4 = \tilde{f}_4(f_{\rm sing})^{2p-1}, \tag{37}$$

rather than by Eqs. (31); the estimate of the parameter c_0 in Eq. (29) is illustrated in Fig. 1b. It is easily seen that the actual interval of $\bar{g}$ values is much narrow than that in the leading logarithmic approximation (where the parameter $1/A_0$ is similar to $\bar{g}$). The optimum $\bar{g}$ values for various functions cover the range of 0.85–1.06, which provides the estimate

$$c_0 = 36.3 \pm 1.8 \tag{38}$$

in good agreement with a theoretical value of 35.09. The exact c_0 value is obtained at $\bar{g} \approx 1.02$ (see Fig. 1b). Finally, Table 4 presents the maximally accurate estimate of the critical point

$$\kappa_c = 0.074840–0.074867, \tag{39}$$

which is available with the existing information. The values accepted below are $\kappa_c = 0.074850$ from the middle of interval (39) and $\bar{g} = 1.020385$, which ensures the exact c_0 value for the [3/3] approximant.

4. RESULTS FOR RENORMALIZATION GROUP FUNCTIONS

The derivatives of singular functions can be written in the form

$$[\ln f_{\mathrm{sing}}]' = \frac{u_1(\tau)}{\kappa_c\tau}, \quad [\ln f_{\mathrm{sing}}]'' = \frac{u_2(\tau)}{(\kappa_c\tau)^2},$$
$$[\ln h_{\mathrm{sing}}]' = \frac{v_1(\tau)}{\kappa_c\tau}, \quad [\ln h_{\mathrm{sing}}]'' = \frac{v_2(\tau)}{(\kappa_c\tau)^2}, \tag{40}$$

where

$$u_1(\tau) = \frac{\bar{g}}{f_{\mathrm{sing}}}\left(1 + \frac{s\bar{g}}{1-\bar{g}\ln\tau}\right),$$
$$u_2(\tau) = \frac{\bar{g}}{f_{\mathrm{sing}}}\left\{1 + \frac{s\bar{g}}{1-\bar{g}\ln\tau} - \frac{s\bar{g}^2}{(1-\bar{g}\ln\tau)^2}\right\} - u_1(\tau)^2,$$
$$v_1(\tau) = -\frac{q\bar{g}^2}{f_{\mathrm{sing}}(f_{\mathrm{sing}}+q\bar{g})}\left(1 + \frac{s\bar{g}}{1-\bar{g}\ln\tau}\right), \tag{41}$$
$$v_2(\tau) = -\frac{q\bar{g}^2}{f_{\mathrm{sing}}(f_{\mathrm{sing}}+q\bar{g})}$$
$$\times\left\{1 + \frac{s\bar{g}}{1-\bar{g}\ln\tau} - \frac{s\bar{g}^2}{(1-\bar{g}\ln\tau)^2}\right.$$
$$\left. - \left(\frac{\bar{g}}{f_{\mathrm{sing}}} + \frac{\bar{g}}{f_{\mathrm{sing}}+q\bar{g}}\right)\left(1 + \frac{s\bar{g}}{1-\bar{g}\ln\tau}\right)^2\right\}.$$

Taking into account Eqs. (40), the substitution of Eqs. (33) into Eqs. (18) provides the parametric representation for the renormalization group functions in the form

$$g = \frac{H(\kappa)}{\kappa^2 f_{\mathrm{sing}}}, \quad H(\kappa) = h_4 h_0^2,$$
$$\frac{\beta(g)}{g} = \frac{2\kappa_c\tau(2-\kappa[\ln h_4 h_0^2]') + 2\kappa u_1}{\kappa_c\tau(1-\kappa[\ln h_0]') + \kappa(1+pu_1)},$$
$$\eta(g) = \frac{-2\kappa_c\tau\kappa[\ln h_0 h_2]' - 2\kappa v_1}{\kappa_c\tau(1-\kappa[\ln h_0]') + \kappa(1+pu_1)},$$
$$\eta_2(g) = -2\{(\kappa_c\tau)^2(1-\kappa^2[\ln h_2]'')$$
$$+ [\kappa_c\tau(1+\kappa[\ln h_2]') + \kappa(1+pu_1+v_1)]^2 \tag{42}$$
$$- \kappa^2(1+pu_2+v_2)\}$$
$$\times\{\kappa_c\tau(1-\kappa[\ln h_0]') + \kappa(1+pu_1)\}^{-1}$$
$$\times\{\kappa_c\tau(1+\kappa[\ln h_2]') + \kappa(1+pu_1+v_1)\}^{-1}.$$

Asymptotic expressions (19) are obtained at $\kappa \longrightarrow 0$ irrespective of the form of regular functions, whereas at $\tau \longrightarrow 0$, we have the results

$$g = \frac{2\bar{g}}{\beta_2 f_{\mathrm{sing}}}, \quad \frac{\beta(g)}{g} = \frac{2\bar{g}}{f_{\mathrm{sing}}} + \frac{2(s-p)\bar{g}^2}{f_{\mathrm{sing}}^2},$$
$$\eta(g) = \frac{2q\bar{g}^2}{f_{\mathrm{sing}}^2}, \quad \eta_2(g) = -\frac{2p\bar{g}}{f_{\mathrm{sing}}}, \tag{43}$$

which reproduce the first two terms of the expansion for $\beta(g)$ and the first terms of the expansions for $\eta(g)$ and $\eta_2(g)$ in Eqs. (26).[1] When the terms with τ are neglected, Eqs. (42) provide the regular expansions of the renormalization group functions in g (certainly without the reproduction of correct coefficients), whereas the terms with τ provide the $\exp(-\mathrm{const}/g)$ singularity, which should exist owing to the factorial divergence of the perturbation series [20, 21]. Thus, the parametric representation is quite "intelligent" and ensures the correct analytical properties at $g \longrightarrow 0$.

The accuracy of the entire construction is determined by the accuracy of the determination of the regular functions $h_i(\kappa)$; the expansions of these functions in κ are obtained from Eqs. (33) and are used to construct the Padé approximants whose behavior in the $(0, \kappa_c)$ interval is regular, because all singularities have been separated. The resulting regular functions are shown in Fig. 2. All approximants of the functions $H(\kappa)$ and $[\ln h_0(\kappa)]'$ provide almost the same

[1] Note that the coefficients β_2, β_3, δ_2, and ζ_1 exhaust invariant (scheme-independent) information on the renormalization group functions and a further refinement of the procedure (the construction of the next-to-next-to-leading logarithmic approximation, etc.) requires the calculation of the subsequent coefficients for the corresponding lattice regularization.

results; small distinctions for the function $[\ln h_4(\kappa)]'$ are visible near κ_c (see Fig. 2). The situation is less satisfactory with the function $[\ln h_2(\kappa)]'$ for which an increase in the order of the Padé approximants leads to an increase in the deviations from the regular behavior predicted in lower approximants. It is unclear whether the sequence of approximants converges sufficiently or such deviations will further increase. Moreover, these deviations can be artefact due to an incompletely consistent separation of singularities leading to a residual singularity in the function $[\ln h_2(\kappa)]'$ (in the used approximation), which affects higher approximants. In the latter case, the behavior predicted by the [3/3], [2/3], and [3/2] approximants can be more authentic. Fortunately, this dilemma can be resolved using the strong-coupling expansions (see Section 5), which certainly indicate that the use of higher Padé approximants is correct and the results obtained in this case are satisfactory. Appendix B presents the parameters of the approximants used for $H(\kappa)$ and $[\ln h_i(\kappa)]'$, which allow the application of parametric representation (42).

To represent the results, it is convenient to use the so-called natural normalization of the charge, which is obtained by the change $g \longrightarrow (16\pi^2/3)g$ and corresponds to the representation of the interaction term in the form $(16\pi^2/4!)g_0\varphi^4$[2]; in this case, the parameter a in the Lipatov asymptotic expression $ca^N\Gamma(N + b)$ [20, 21] is unity and the nearest singularity in the Borel plane lies at the unit distance from the coordinate origin [21]; this property defines functions varying at an approximately unit scale. The solid lines in Fig. 3 are the resulting renormalization group functions, whereas the dashed lines are the strong- and weak-coupling asymptotic behaviors. The approach to the strong-coupling asymptotics is strongly prolonged in agreement with the results reported in [22]. However, the prolongation of the one-loop behavior of the β function pointed out in that work is not confirmed: it appears to be an artefact, conditioned by essential exceeding of the limiting value of $\beta(g)/g$ obtained in [22] in comparison with Fig. 3 [1].

To illustrate the accuracy of the construction, the results obtained when the functions $h_i(\kappa)$ were changed to constants are given by the dotted lines; in this case, the results contain no information on these functions, because $[\ln h_i]' = 0$ and a constant value of $H(\kappa)$ is fixed by Eq. (29). It is easy to see that an accuracy of about 1% for $\beta(g)/g$ and $\eta_2(g)$ is reached even in the complete absence of information on regular functions.[3] The real uncertainty of the construction is about two orders of magnitude smaller than the difference between the solid and dotted lines, because the regular functions (see Fig. 2) are specified better than 10^{-2} expect for the region $\kappa > 0.8\kappa_c$, where the error for the function $[\ln h_2]'$ can reach 10%. However, this region corresponds to $g < 0.5$ (see Fig. 4), where the effect of regular functions is insignificant.

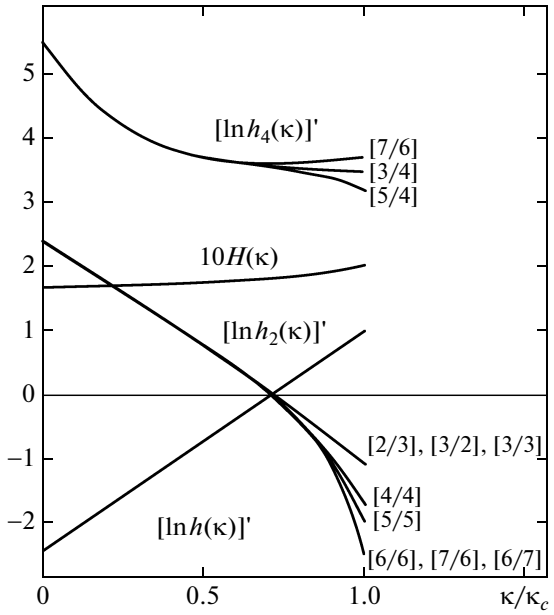

Fig. 2. Regular functions $H(\kappa)$ and $[\ln h_i(\kappa)]'$ obtained in the Padé approximation.

5. STRONG-COUPLING EXPANSIONS

Expanding the right-hand sides of Eqs. (18) in κ and expressing κ in terms of g, it is easy to verify that the functions $\beta(g)/g$, $\eta(g)$, and $\eta_2(g)$ are expanded in $g^{-2/d}$ as

$$\frac{\beta(g)}{g} = \sum_{N=0}^{\infty} B_N(-g^{-2/d})^N \text{ etc.} \qquad (44)$$

The expansion coefficients up to $N = 13$ recalculated from high-temperature series are given in Table 5.[4] It

[2] The traditional representation $g_0\varphi^4/8$ in the n-component case is motivated by the fact that the vertex $\Gamma^{(4)}_{\alpha\beta\gamma\delta} = gI_{\alpha\beta\gamma\delta}$ in the lowest order is $g_0 I_{\alpha\beta\gamma\delta}$, which ensures the relation $g = g_0$ in the limit $g_0 \longrightarrow 0$. In the scalar case, the tensor $I_{\alpha\beta\gamma\delta}$ is reduced to three, and the interaction is represented as $g_0\varphi^4/4!$. This motivation logical at first glance is illusory, because the bare charge has no physical meaning.

[3] The reason is that the term $[\ln h_i]'$ in Eqs. (42) has the factor $\kappa\kappa_c\tau = \kappa(\kappa_c - \kappa)$, which is small both for $\kappa \longrightarrow 0$ and for $\kappa \longrightarrow \kappa_c$; this factor in the middle of the interval $\kappa = \kappa_c/2$ is equal to $\kappa_c^2/4$, whereas the other terms are on the order of κ_c. In view of $\kappa_c \approx 1/15$, the effect of regular functions on $\beta(g)/g$ and $\eta_2(g)$ is about 1%. The situation for $\eta(g)$ is different in view of the absent of the $\kappa_c\tau$ term in the numerator.

[4] Fourteen digits output by a computer are formally presented. The accuracy decreases beginning with $N = 3$ and the last four digits are unreliable at $N = 13$.

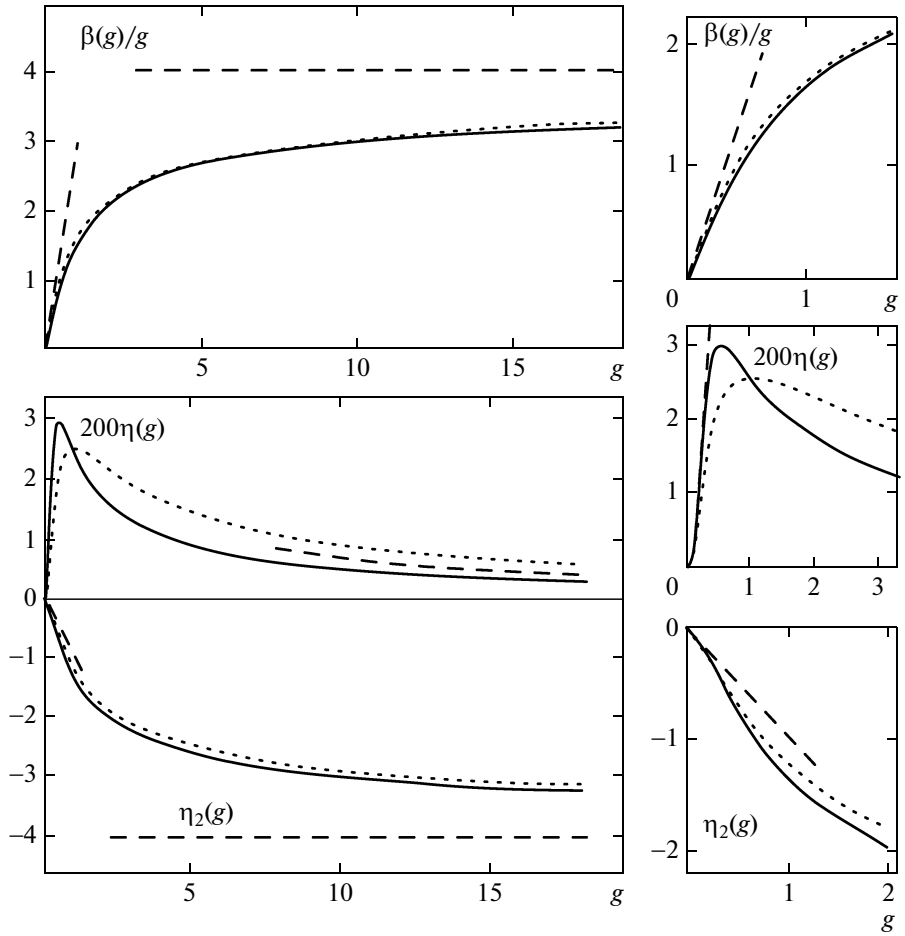

Fig. 3. Solid lines are the renormalization group functions. The dashed lines are the strong- and weak-coupling asymptotic behaviors. The dotted lines are the results obtained under the assumption of the constancy of regular functions $h_i(\kappa)$ under which Eqs. (42) contain no information on them.

is easy to verify that the ratios B_{N+1}/B_N are the same order of magnitude for all N values, indicating the finite convergence radius. The Padé analysis of series (44) reveals poles in the region $|g^{-1/2}| \sim 0.1$; these poles for most approximants do not lie on the positive semiaxis in agreement with the regularity of the renormalization group functions. To obtain the correct power-law behavior in the limit $g \longrightarrow 0$, it is necessary to use the $[N/N+2]$ approximants for $\beta(g)/g$ and $\eta_2(g)$ and the $[N/N+4]$ approximants for $\eta(g)$. Such a procedure predicts δ_2 with an accuracy of about 20%, whereas β_2 and ζ_1 are estimated only by the order of magnitude. For this reason, the summation of series (44) in the region of small g values gives less accurate results than the procedure described above.

All approximants provide almost coinciding results in the region of large g values; this coincidence with an accuracy of about 1% holds to $g = 0.5$ (in the natural normalization). Such estimates for the functions $\beta(g)$ and $\eta_2(g)$ are in agreement with the more accurate results obtained above. These estimates for the function $\eta(g)$ certainly indicate that the highest-order approximants should be used for $[\ln h_2]'$ and the results are confirmed at a level of about 1%. Series (44) can apparently be used more efficiently, but analysis of this possibility is beyond the scope of this work.

6. DISCUSSION OF THE RESULTS

The resulting β function is non-alternating and has the asymptotic behavior $\beta(g) = 4g$ in the $\lim g \longrightarrow \infty$. According to the classification proposed by Bogoliubov and Shirkov [23] (see discussion in [1]), this means the possibility of the construction of a continual theory with a finite interaction at large distances. The last conclusion contradicts the widespread opinion that the φ^4 theory is "trivial" [24–28]. As was discussed in [1, 30], two definitions—first, Wilson triviality [24] and, second, mathematical triviality [25, 26]—were confused in the literature. The first triviality is firmly established (it corresponds to the positivity of the β function), whereas pieces of evidence in favor of the second triviality are scarce [27] and allow another interpretation [1, 30]. According to the above analysis, we have no contradictions in the properties of

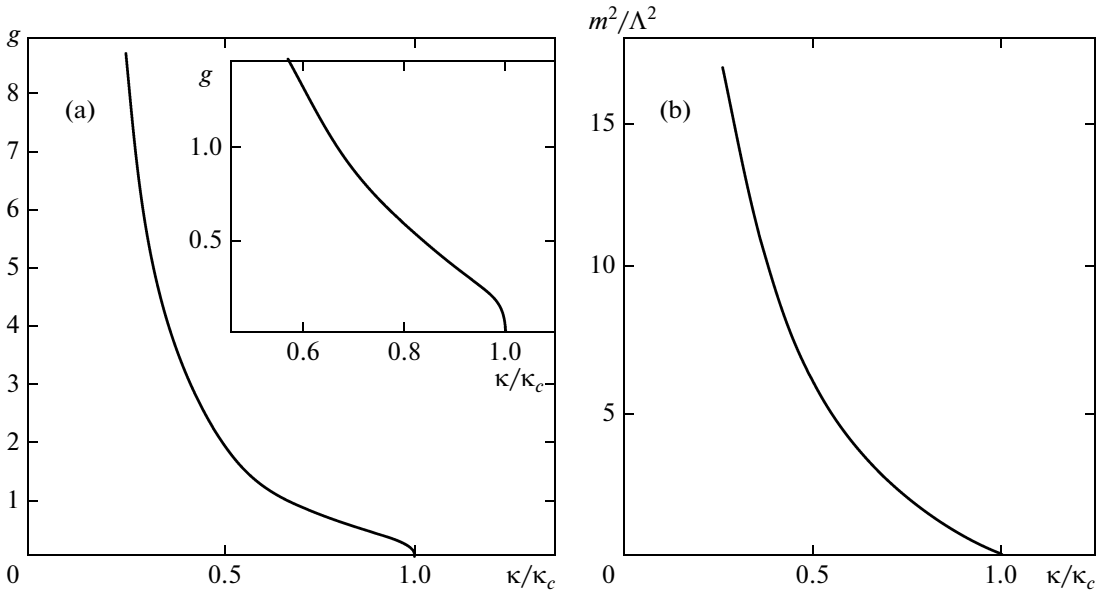

Fig. 4. Renormalized (a) charge g and (b) mass m versus κ/κ_c.

the lattice the φ^4 theory with the works cited in [1, 30]. However, there is a conceptual contradiction concerning the role and significance of the lattice theory.

The usual point of view implies that the lattice φ^4 theory provides a reasonable approximation for the actual field theory. This interpretation provides the natural condition $\xi \gg a$ according to which many sites of the lattice should be at the characteristic variation scale of the field $\varphi(x)$. This condition can be weakened to the condition $\xi \gtrsim a$ or forced to $\xi/a \longrightarrow \infty$. In the former case, the restriction $g \lesssim 1$ is obtained for the renormalized charge (in the natural normalization) [28], while $g = 0$ in the latter case (corresponding to the phase transition point). Thus, the usual statements are obtained: the theory trivial in the continual limit ($\Lambda/m \longrightarrow \infty$), whereas the interaction in the presence of a cutoff is limited from above and cannot be strong. The latter circumstance is used to obtain an upper bound for the mass of the Higgs boson [28, 29].

Our position is that the lattice theory should not be considered as any approximation to the actual theory (although this is possible at $g_0 \ll 1$). The continual theory fundamentally involves no lattice; a lattice appears only in the bare theory, which is an auxiliary construction and is then completely eliminated. The bare theory has no physical meaning and should not satisfy any physical requirements. Without the restriction $\xi \gtrsim a$, the renormalized charge can have any value (see Fig. 4). The proposed concept is completely consistent with the "rules" accepted in mathematical works [25, 26] according to which the continual limit $a \longrightarrow 0$ is taken at arbitrarily chosen dependences $g_0(a)$ and $m_0(a)$; in this work, they are taken under conditions (5).

The only alternative for the perturbative approach is that all quantities referring to the continual theory are expressed in terms of functional integrals. These integrals depend on g_0, m_0, and Λ and, with dimensionality taken into account we have for the charge, mass, and other physical quantities A_i (observables, RG functions, etc.), the results are

$$g = F_g(g_0, m_0/\Lambda), \quad m = \Lambda F_g(g_0, m_0/\Lambda), \qquad A_i = \Lambda^{d_i} F_i(g_0, m_0/\Lambda), \tag{45}$$

where d_i is the physical dimension of the quantity A_i. According to Eqs. (45), the real designation of the bare theory is to ensure the representation of the quantities of interest in a parametric form. The relations between g, m, and A_i are of physical interest; the parametric representation is of no deep sense in view of its ambiguity: it can be written in various forms by changing g_0 and m_0/Λ to any other pair of variable. For this reason, an attempt to give the physical sense to the bare theory faces the question: Why one of numerous parameterizations is of particular significance?

Excluding g_0 and m_0/Λ in favor of g and m/Λ, it is possible to arrive at the relation

$$A_i = m^{d_i} \tilde{F}_i(g, m/\Lambda). \tag{46}$$

In the general case, the exclusion of the dependence on Λ requires the passage to the limit $m/\Lambda \longrightarrow 0$, which corresponds to the critical point and returns us to the "zero charge." However, the central point is that the general-position situation does not occur in Eq. (46): after the transformation to the Ising model (valid under conditions (5)), all functions in Eqs. (45)

Table 5. Coefficients of the expansions of the functions $\beta(g)/g$, $\eta(g)$, and $\eta_2(g)$

N	$\beta(g)/g$	$\eta(g)$	$\eta_2(g)$
0	4.0000000000000	0.0000000000000	−4.0000000000000
1	−26.127890589687	0.0000000000000	26.127890589687
2	106.66666666666	1.7777777777777	−60.444444444444
3	−557.39499924665	−11.612395817638	81.286770723472
4	3214.2222222221	29.708641975308	−44.879012345695
5	−16396.702894504	22.708685154477	−1208.7213779957
6	67356.444444432	−961.13125612398	9071.1992161454
7	−139720.34647768	7188.4949076856	−49662.878604241
8	−717634.37037244	−27680.892323840	197619.39191503
9	9878174.8209247	−7609.7703277375	−226822.08364126
10	−59767955.489704	938372.27840847	−3873286.8465521
11	186179701.36334	−7226487.6363735	41826925.334797
12	355069103.58896	27981910.625966	−249549251.38460
13	−8851453360.7421	7407298.5714308	794136522.54618

depend on the single parameter κ; as a result, the dependence on m/Λ is completely absent in Eq. (46):[5]

$$A_i = m^{d_i} F_i(g). \tag{47}$$

The renormalization program is thereby completed and no additional passages to limits are required. This means that (a) the lattice can be retained in the bare theory (as a convenient technical tool for the representation of functional integrals) and (b) the relation between m and Λ can be assumed to be arbitrary, which ensures the attainability of any g value (see Fig. 4).

The above procedure is a real scheme for constructing the continual φ^4 theory with a finite interaction. In fact, the dependences of g and m on bare parameters (see Fig. 4), as well as the results for the renormalization group functions (see Fig. 3), have been obtained in this paper.

APPENDIX A

Next-to-Leading Logarithmic Approximation

The basic formulas referring to the next-to-leading logarithmic approximation underlying representation (33) will be given below. The starting point is the Callan–Symanzik equation in the cutoff scheme[6]

$$\left[\frac{\partial}{\partial \ln\Lambda} + \beta(g_0)\frac{\partial}{\partial g_0} - \gamma(g_0)\right] F\left(g_0, \frac{\Lambda}{m}\right) = 0, \tag{A.1}$$

where the function F satisfies the logarithmic expansion

$$F\left(g_0, \frac{\Lambda}{m}\right) = \sum_{N=0}^{\infty} g_0^N \sum_{K=0}^{N} A_N^K \left(\ln\frac{\Lambda}{m}\right)^K. \tag{A.2}$$

The substitution of Eq. (A.2) to Eq. (A.1) taking into account the expansions

$$\beta(g_0) = \sum_{M=2}^{\infty} \beta_M g_0^M, \quad \gamma(g_0) = \sum_{M=1}^{\infty} \gamma_M g_0^M$$

yields the system of recurrence relations for the coefficients A_N^K:

$$-KA_N^K = \sum_{M=1}^{N-K+1} [\beta_{M+1}(N-M) - \gamma_M] A_{N-M}^{K-1}, \tag{A.3}$$
$$K = 1, 2, \ldots, N.$$

In particular, for K close to N,

$$-NA_N^N = [\beta_2(N-1) - \gamma_1] A_{N-1}^{N-1}, \tag{A.4}$$

$$-(N-1)A_N^{N-1} = [\beta_2(N-1) - \gamma_1] A_{N-1}^{N-2} + [\beta_3(N-2) - \gamma_2] A_{N-2}^{N-2},$$

$$-(N-2)A_N^{N-2} = [\beta_2(N-1) - \gamma_1] A_{N-1}^{N-3} + [\beta_3(N-2) - \gamma_2] A_{N-2}^{N-3}$$

[5] This is not surprising, because the passage to the continual limit was performed in the process of the transformation to the Ising model [2], which was required by the needs of renormalized (not bare) theory.

[6] Its difference from Eq. (10) in this case is of no significance, because the first coefficients β_2, β_3, δ_2, and ζ_1 are independent of the renormalization scheme.

$$+ [\beta_4(N-3) - \gamma_3]A_{N-3}^{N-3}$$

etc. The first equation in Eqs. (A.4) is solved directly; after that, the next equations can be solved one-by-one using the method of variation of constants.

Vertex $\Gamma^{(1,2)}$. For this vertex, $\gamma(g_0) = \eta_2(g_0)$, all coefficients A_N^N are nonzero, and $A_0^0 = 1$; the first two equations in Eqs. (A.4) give

$$\begin{aligned} A_N^N &= (-\beta_2)^{N-1}\frac{\Gamma(N+p)}{\Gamma(p)\Gamma(N+1)}, \\ p &= -\frac{\gamma_1}{\beta_2} = -\frac{\zeta_1}{\beta_2}, \\ A_N^{N-1} &= (-\beta_2)^{N-1}\frac{\Gamma(N+p)}{\Gamma(1+p)\Gamma(N)} \\ &\times\left\{p\frac{\beta_3}{\beta_2}\sum_{n=1}^{N-1}\frac{1}{n+p} + O(1)\right\}. \end{aligned} \tag{A.5}$$

The substitution of Eqs. (A.5) into Eq. (A.2) and the summation of the corresponding series using the formulas

$$\begin{aligned} (1+x)^\alpha &= \sum_{n=0}^{\infty}\frac{\Gamma(n-\alpha)}{\Gamma(-\alpha)\Gamma(n+1)}(-x)^n, \\ &(1+x)^\alpha\ln(1+x) \\ &= \sum_{n=0}^{\infty}\frac{\Gamma(n-\alpha)}{\Gamma(-\alpha)\Gamma(n+1)}(-x)^n\sum_{k=0}^{n-1}\frac{1}{\alpha-k} \end{aligned} \tag{A.6}$$

yield

$$\begin{aligned} \Gamma^{(1,2)} = \Bigg\{1 &+ O(g_0) + \beta_2 g_0\ln\frac{\Lambda}{m} \\ &+ g_0\frac{\beta_3}{\beta_2}\ln\left(1+\beta_2 g_0\ln\frac{\Lambda}{m}\right)\Bigg\}^{-p}. \end{aligned} \tag{A.7}$$

The $O(g_0)$ terms will be omitted below.

The renormalized charge g satisfies Eq. (A.1) with $\gamma(g_0) \equiv 0$, whereas all coefficients A_N^N in expansion (A.2) are zero and $A_1^0 = 1$. Similar to Eqs. (A.5) and (A.7),

$$\begin{aligned} A_N^{N-1} &= (-\beta_2)^{N-1}, \\ A_N^{N-2} &= (-\beta_2)^{N-2}(N-1) \\ &\times\left\{\frac{\beta_3}{\beta_2}\sum_{n=1}^{N-1}\frac{1}{n} + O(1)\right\}, \end{aligned} \tag{A.8}$$

we have a result

$$\begin{aligned} g = g_0\Bigg\{1 &+ \beta_2 g_0\ln\frac{\Lambda}{m} \\ &+ g_0\frac{\beta_3}{\beta_2}\ln\left(1+\beta_2 g_0\ln\frac{\Lambda}{m}\right)\Bigg\}^{-1}, \end{aligned} \tag{A.9}$$

which can also be obtained directly from the Gell-Mann–Low equation.

Renormalized mass. Neglecting the Z factor, the Ward identity

$$\Gamma^{(1,2)} = \frac{d}{dm_0^2}\Gamma^{(0,2)} = \frac{d}{dm_0^2}\frac{m^2}{Z} \tag{A.10}$$

can be written in the form $dm_0^2/dm^2 = 1/\Gamma^{(1,2)}$; the integration with respect to m^2 within the necessary accuracy is reduced to the multiplication by m^2,

$$\begin{aligned} m^2 = (m_0^2 - m_c^2)\Bigg\{1 &+ \beta_2 g_0\ln\frac{\Lambda}{m} + g_0\frac{\beta_3}{\beta_2} \\ &\times\ln\left(1+\beta_2 g_0\ln\frac{\Lambda}{m}\right)\Bigg\}^{-p}, \end{aligned} \tag{A.11}$$

where m_c^2 is the m_0^2 value corresponding to the transition point. The introduction of the dimensionless distance to the transition $\tau \propto (m_0^2 - m_c^2)$ and iterative exclusion of m from the right-hand side give

$$\begin{aligned} m^2 &= \tau\left[1+\bar g\ln\frac{1}{\tau} + s\bar g\ln\left(1+\bar g\ln\frac{1}{\tau}\right)\right]^{-p}, \\ \bar g &= \frac{\beta_2 g_0}{2}, \end{aligned} \tag{A.12}$$

where s is given in Eqs. (36). Similarly, Eq. (A.9) reduces to the form

$$\begin{aligned} g = \frac{2}{\beta_2} \\ \times\bar g\left\{1+\bar g\ln\frac{1}{\tau} + s\bar g\ln\left(1+\bar g\ln\frac{1}{\tau}\right)\right\}^{-1}. \end{aligned} \tag{A.13}$$

The Z *factor* satisfies Eq. (A.1) with $\gamma(g_0) = -\eta(g_0)$, and in expansion (A.2), $A_0^0 = 1$, $A_1^0 = A_1^1 = 0$, and $A_N^N = 0$ for $N \geq 2$. Similar to Eqs. (A.8),

$$\begin{aligned} A_N^{N-1} &= A_2^1(-\beta_2)^{N-2}, \\ A_N^{N-2} &= A_2^1(-\beta_2)^{N-2}(N-1) \\ &\times\left\{-\frac{\beta_3}{\beta_2^2}\sum_{n=2}^{N-1}\frac{1}{n} + O(1)\right\}, \end{aligned} \tag{A.14}$$

for $N \geq 2$; the summation gives

Table 6. Parameters of Padé approximation (24) of regular functions

	$H(\kappa)$		$[\ln h_0(\kappa)]'$	
n	p_n	q_n	p_n	q_n
0	0.166666	1.000000	−2.389114	1.000000
1	2.173343	12.28756	39.93594	1.218909
2	−8.874246	−6.056224	134.2565	−14.76806
3	103.5876	−124.8396	−1759.943	498.1762
4	0	0	14434.97	−2468.179
	$[\ln h_2(\kappa)]'$		$[\ln h_4(\kappa)]'$	
n	p_n	q_n	p_n	q_n
0	2.416517	1.000000	5.530725	1.000000
1	−50.63241	−3.794992	13.37787	21.09480
2	−345.9676	−201.7335	630.6971	57.28333
3	9156.772	738.3887	3430.220	252.1934
4	−1285.833	4787.275	0	10511.06
5	−267488.9	−26827.13	0	0
6	109199.7	363530.4	0	0

$$Z = 1 + \frac{A_2^1 g_0}{\beta_2} - \frac{A_2^1 g_0}{\beta_2}\left\{1 + \beta_2 g_0 \ln\frac{\Lambda}{m} + g_0\frac{\beta_3}{\beta_2}\ln\left(1 + \beta_2 g_0 \ln\frac{\Lambda}{m}\right)\right\}^{-1}. \quad \text{(A.15)}$$

Taking into account the relation $A_2^1 = -\delta_2$, expressing m in terms of τ, and omitting an insignificant constant factor, with the necessary accuracy one obtains

$$Z = 1 + \frac{2\delta_2}{\beta_2^2}\bar{g}\left\{1 + \bar{g}\ln\frac{1}{\tau} + s\bar{g}\ln\left(1 + \bar{g}\ln\frac{1}{\tau}\right)\right\}^{-1}. \quad \text{(A.16)}$$

The substitution of Eqs. (A.12), (A.13), and (A.16) into the relations

$$\frac{K_2}{\tilde{K}_2} = m^2, \quad \frac{K_2}{K_0} = \frac{Z}{m^2}, \quad \frac{K_4 K_0}{K_2^2} = -\frac{g}{m^4}, \quad \text{(A.17)}$$

yields Eqs. (33) for $f_i(\kappa)$. The difference of the Z factor from unity corresponds to the corrections of the $g_0/\ln\tau$ order, which were neglected above, and is strictly speaking beyong accuracy. However, without the inclusion of the Z factor, the product $f_0 f_2$ would be a regular function and, correspondingly, the behavior of $\eta(g)$ at small g values would be incorrect. For this reason, the function h_{sing} corresponding to the Z factor is introduced in Eqs. (33) by the minimal manner to ensure the correct singularity in $f_0 f_2$.

APPENDIX B

Padé Approximation of Regular Functions

Table 6 shows the coefficients p_n and q_n in Eq. (24) for the Padé approximation of the regular functions $H(\kappa)$ and $[\ln h_i(\kappa)]'$; the lowest-order approximants having the complete accuracy are presented. The singularities were separated with the values $\kappa_c = 0.074850$ and $\bar{g} = 1.020385$.

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