To separation of variables in the Fokker–Plank equations

I. M. Suslov
P.L.Kapitza Institute for Physical Problems, 119334 Moscow, Russia
E-mail: suslov@kapitza.ras.ru

It is well-known, that for separation of variables in the eigenvalue problem, the corresponding operator should be represented as a sum of operators depending on single variables. In the case of the Fokker–Plank equations, separation of variables is possible under essentially weaker conditions.

For separation of variables in the eigenvalue problem

$$\hat{L}\, P(x,y) = \lambda\, P(x,y) \tag{1}$$

the operator $\hat{L}$ should be represented as a sum of two operators $\hat{L}_x + \hat{M}_y$, depending only on x and y correspondingly.

Conditions for separation of variables in the Fokker–Plank equations appear to be essentially weaker. For example, for the equation describing the time evolution of the probability distribution $P \equiv P(x,y)$,

$$\frac{\partial P}{\partial t} = \left\{ \hat{L}_{x,y} P \right\}'_x + \left\{ \hat{M}_y P \right\}'_y , \tag{2}$$

it is sufficient that the operator $\hat{M}_y$ in the last term depends only on y, while the operator $\hat{L}_{x,y}$ remains arbitrary. Indeed, setting $P = P(x)P(y)$ and dividing by $P(x)$, one has

$$-\frac{\partial P(y)}{\partial t} + \left\{ \hat{M}_y P(y) \right\}'_y = \\ = \frac{P(y)}{P(x)} \frac{\partial P(x)}{\partial t} - \frac{1}{P(x)} \left\{ \hat{L}_{x,y} P \right\}'_x . \tag{3}$$

The left-hand side is independent of x, and can be considered as a certain function $F(y)$. Then

$$P(y) \frac{\partial P(x)}{\partial t} - \left\{ \hat{L}_{x,y} P \right\}'_x = F(y) P(x) \tag{4}$$

and integration over x gives $F(y) \equiv 0$, since the left-hand side turns to zero, while the integral over $P(x)$ is equal to unity due to normalization. As a result, left-hand side and right-hand side of Eq. 3 turn to zero independently, and the equation for $P(y)$ is separated

$$\frac{\partial P(y)}{\partial t} - \left\{ \hat{M}_y P(y) \right\}'_y = 0 . \tag{5}$$

On the other hand, integrating (3) over y, one has

$$\frac{\partial P(x)}{\partial t} - \left\{ \hat{\mathcal{L}}_x P(x) \right\}'_x = 0 , \tag{6}$$

where

$$\hat{\mathcal{L}}_x = \int \hat{L}_{x,y} P(y) dy . \tag{7}$$

The given considerations are very general and applicable to any diffusion-type equation: the right-hand side of the latter is always a sum of full derivatives, in order to provide the conservation of probability. As a result, the conditions for separation of variables appear to be always weaker than for equation (1). In our opinion, this fact should be mentioned in any courses of the mathematical physics; unfortunately, it is not the case.

The separation of variables in the Fokker–Plank equations was discussed in the comparatively new papers (e.g. [1, 2, 3]), but under rather restricted assumptions. The equation of type (2) arises in the theory of 1D localization, where it describes the evolution of the mutual distribution $P(\rho,\psi)$ of the Landauer resistace ρ and the phase variable $\psi = \theta - \varphi$, where θ and φ are phases entering the transfer matrix (see Eq.28 in [4] and the comments after it). Analogous situation is expected in description of quasi-1D systems in the framework of the generalized version [5] of the Dorokhov–Mello–Pereyra-Kumar equation [6, 7]. It looks probable that analogous equations arose in other fields of physics and in some cases the fact of separation of variables was revealed by the corresponding authors. However, the general character of this result was not emphasized, and it remains unknown to wide audience.

References

[1] A. Andreitsev, Symmetry in Nonlinear Mathematical Physics **1**, 211, (1997).

[2] A. Zhalij, J. Phys. A: Math. Gen. **32**, 7393 (1999).

[3] W. Rui, X. Yang, F. Chen, Physica A **595**, 127068 (2022).

[4] I. M. Suslov, Phil. Mag. Lett., https://doi.org/10.1080/09500839.2022.2096267; arXiv: 2110.12986.

[5] I. M. Suslov, J. Exp. Theor. Phys. **127**,131 (2018) [Zh. Eksp. Teor. Fiz. **154**, 152 (2018)].

[6] O. N. Dorokhov, JETP Letters **36**, 318 (1982) [Pis'ma Zh. Eksp. Teor. Fiz. **36**, 259 (1979)].

[7] P. A. Mello, P. Pereyra, N. Kumar, Ann. Phys. (N.Y.) **181**, 290 (1988).