

Possibility of the 2D Anderson Transition and Generalized Lyapunov Exponents

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The possible existence of the Anderson transition in 2D systems without interaction and spin-orbit effects (such as the usual Anderson model) becomes recently a subject of controversy in the literature [1, 2, 3]. Comparative analysis of approaches based on generalized Lyapunov exponents is given, in order to resolve controversy.

1. Introduction

Possible existence of the Anderson transition in the 2D case becomes recently a subject of controversy in the literature. Kuzovkov et al [1] studying a growth of the second moments for a particular solution of the quasi-1D Schroedinger equation and interpreting results in terms of the signal theory came to conclusion that the first order Anderson transition exists in the usual 2D Anderson model. This incredible result was analyzed by Markos et al [2] and the opposite conclusion was drawn: behavior of the second moments is qualitatively different from found in [1] and contains no evidence of the metallic phase. On the other hand, the present author [3] used the analogous approach and came to conclusion on existence of the Kosterlitz-Thouless type transition, which however can be realized not in all 2D systems.

The present paper has an aim to analyze the existing controversy and is organized in the following manner. In Sec. 2 we give brief exposition of paper [3], which we consider as the proper treatment of the problem. Approach by Kuzovkov et al [1] is discussed in Sec. 3: the results for the second moments are shown to be partially correct but their interpretation is not satisfactory. Section 4 deals with the approach advanced by Markos et al [2]: their statements are shown to be related with improper calculation of the matrix product. Conclusions are made in Sec. 5.

2. Second moments of the Cauchy solution and their relation with the Anderson transition

Consider the 2D Anderson model described by the discrete Schroedinger equation

$$\psi_{n+1,m} + \psi_{n-1,m} + \psi_{n,m+1} + \psi_{n,m-1} + V_{n,m}\psi_{n,m} = E\psi_{n,m} \quad (1)$$

and interpret it as a recurrence relation in the variable n , which we accept as a longitudinal coordinate. Initial conditions are assumed to be fixed on the left end of the system, while

the periodic boundary conditions are accepted in the transverse direction, $\psi_{n,m+L} = \psi_{n,m}$. Site energies $V_{n,m}$ are considered as uncorrelated random quantities with the first two moments

$$\langle V_{n,m} \rangle = 0, \quad \langle V_{n,m} V_{n',m'} \rangle = W^2 \delta_{n,n'} \delta_{m,m'}. \quad (2)$$

The growth of the second moments for this problem can be studied using the old idea by Thouless [4] based on the observation that variables $\psi_{n,m}$ are statistically independent of $V_{n,m}$ with the same n . The main quantity of interest is $\langle \psi_{n,m}^2 \rangle$; solving (1) for $\psi_{n+1,m}$ and averaging its square, we can relate it with the pair correlators containing lower values of n . Deriving analogous equations for the pair correlators, we end with the closed system of difference equations for the quantities

$$\begin{aligned} x_{m,m'}(n) &\equiv \langle \psi_{n,m} \psi_{n,m'} \rangle, \\ y_{m,m'}(n) &\equiv \langle \psi_{n,m} \psi_{n-1,m'} \rangle, \\ z_{m,m'}(n) &\equiv \langle \psi_{n-1,m} \psi_{n,m'} \rangle, \end{aligned} \quad (3)$$

which for $E = 0$ has a form [3]

$$\begin{aligned} x_{m,m'}(n+1) &= W^2 \delta_{m,m'} x_{m,m'}(n) + x_{m+1,m'+1}(n) + x_{m-1,m'+1}(n) + x_{m+1,m'-1}(n) + x_{m-1,m'-1}(n) + \\ &\quad + x_{m,m'}(n-1) + y_{m+1,m'}(n) + y_{m-1,m'}(n) + z_{m,m'+1}(n) + z_{m,m'-1}(n) \\ y_{m,m'}(n+1) &= -x_{m+1,m'}(n) - x_{m-1,m'}(n) - z_{m,m'}(n) \\ z_{m,m'}(n+1) &= -x_{m,m'+1}(n) - x_{m,m'-1}(n) - y_{m,m'}(n). \end{aligned} \quad (4)$$

This is a set of the linear difference equations with independent of n coefficients and its solution is exponential in n [6]

$$x_{m,m'}(n) = x_{m,m'} e^{\beta n}, \quad y_{m,m'}(n) = y_{m,m'} e^{\beta n}, \quad z_{m,m'}(n) = z_{m,m'} e^{\beta n}. \quad (5)$$

The formal change of variable is useful

$$x_{m,m'} \equiv \tilde{x}_{m,m'-m} = \tilde{x}_{m,l}, \quad \text{etc.}, \quad (6)$$

where $l = m' - m$. Substitution of (6) and (5) to (4) gives the difference equations whose coefficients contain no m dependence and their solution is exponential in m

$$\tilde{x}_{m,l} = x_l e^{i p m}, \quad \text{etc.}, \quad (7)$$

where allowed values for p are determined by the periodical boundary conditions in the transverse direction. Excluding $y_{m,l}$ and $z_{m,l}$ from the first equation in (40), we end with the equation

$$\begin{aligned} x_{l+2} e^{-i p} + x_{l-2} e^{i p} + V \delta_{l,0} x_l &= \epsilon x_l, \quad x_{l+L} = x_l, \\ \epsilon &= 2 \cosh \beta, \quad V = \frac{W^2 \sinh \beta}{\cosh \beta - \cos p} \end{aligned} \quad (8)$$

describing a single impurity in a periodic chain. The positive exponents β_s for finite odd L are determined by equation

$$2(\cosh \beta_s - \cos p_s) = W^2 \coth(\beta_s L/2), \quad p_s = 2\pi s/L, \quad s = 0, 1, \dots, L-1. \quad (9)$$

Allowed values of p_s and β_s become dense in the large L limit, and the quantities β and p can be considered as continuous; the minimal value of β is realized for $p = \pi$ and can be easily found in the large L limit

$$\beta_{min} = \begin{cases} \cosh^{-1}(W^2/2 - 1), & W^2 > 4 \\ \frac{2}{L} \tanh^{-1}(W^2/4), & W^2 < 4 \\ \frac{2 \ln L - 2 \ln \ln L + \dots}{L}, & W^2 = 4. \end{cases} \quad (10)$$

The character of solution is qualitatively changed at $W_c = 2$; this value of disorder corresponds to a singular point found in [1] but its physical sense is not evident.

Interpretation of results can be given using the finite-size scaling approach in the form given by Pichard and Sarma [5]. A quasi-1D system is always localized and the localization length ξ_{1D} can be introduced for it. The knowledge of this length can be used for the study of the Anderson transition in higher-dimensional systems. It is easy to show that $\xi_{1D} \rightarrow const$ in the localized phase and $\xi_{1D}/L \rightarrow \infty$ in the metallic state in the large L limit [3, 5]. If we introduce a scaling parameter

$$g(L) = \frac{\xi_{1D}}{L} \quad (11)$$

then it increases with L in the metallic phase and decreases in the insulator phase. If a scaling relation is postulated

$$g(L) = F\left(\frac{L}{\xi}\right) \quad (12)$$

then $g(L)$ remains constant in the critical point and its general behavior is shown in Fig. 1,a. The relation (12) was never proved but its validity is expected by analogy with the usual phase transition theory.

The solution $\psi_n(r_\perp)$ of the Cauchy problem for the quasi-1D Schroedinger equation with the initial conditions on the left edge allows decomposition

$$\psi_n(r_\perp) = A_1 h_n^{(1)}(r_\perp) e^{\gamma_1 n} + A_2 h_n^{(2)}(r_\perp) e^{\gamma_2 n} + \dots + A_m h_n^{(m)}(r_\perp) e^{\gamma_m n} \quad (13)$$

where r_\perp is the transverse coordinate (like m in (1)), the quantities $h_n^{(s)}(r_\perp)$ have no systematic growth in n , while the Lyapunov exponents γ_s tend to the constant values in the large n limit. The quantity ξ_{1D} can be estimated as inverse of the minimal positive Lyapunov exponent γ_{min} , i.e. $\xi_{1D} \sim 1/\gamma_{min}$; this estimation in combination with the scaling relation (12) is a basis for the most popular of contemporary numerical algorithms (see,

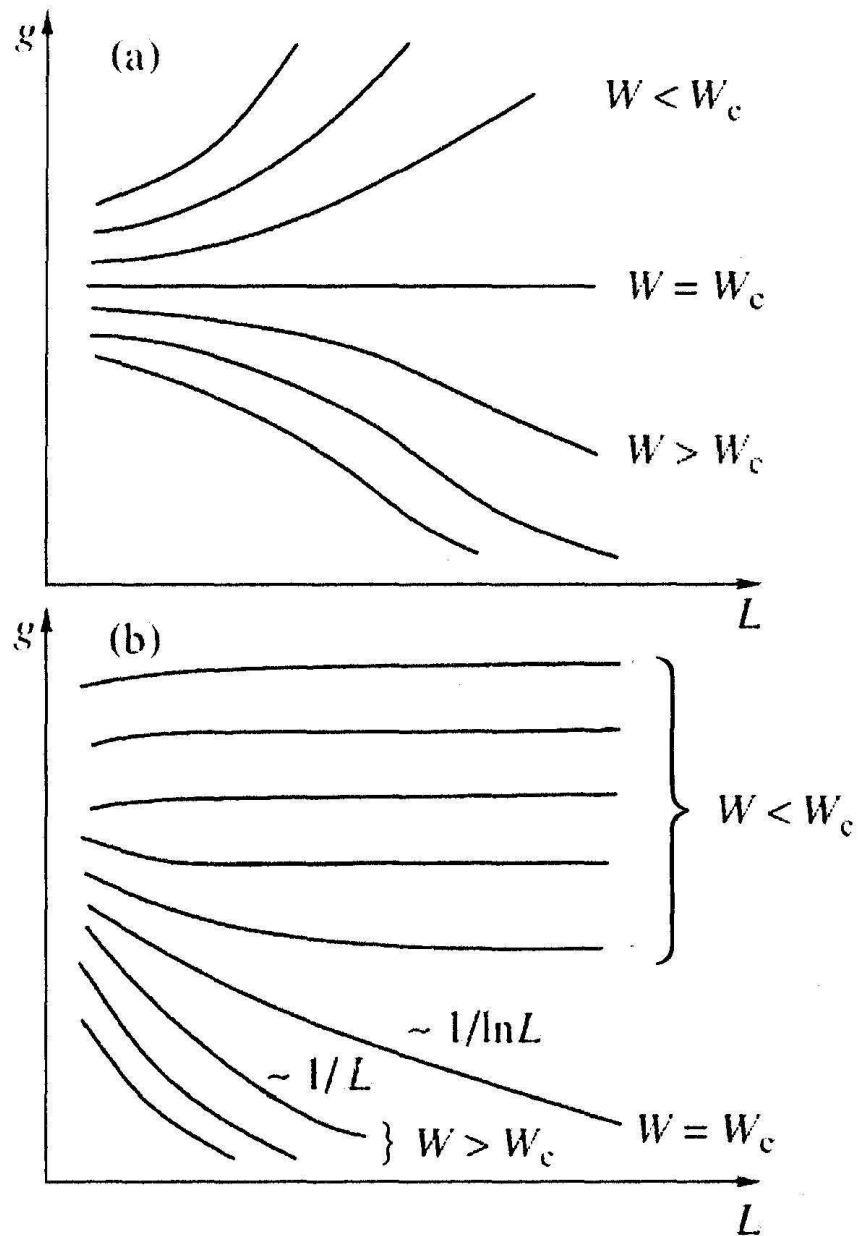


Figure 1: a — Typical behavior of $g(L)$ in the case of one-parameter scaling: different curves for $W > W_c$ or $W < W_c$ can be made coinciding by a scale transformation. b — Behavior of $g(L)$ according to Eq. 9: crude violation of scaling is evident.

e.g. a review article [7]). Due to evident analogy between (5) and (13) we can refer (as suggested in [1]) to the exponents β_s as generalized Lyapunov exponents.

No rigorous relation exists between β_{min} and γ_{min} ; however, these quantities are very close from the physical viewpoint. Indeed [3]: (a) inequality $\beta_{min} \geq 2\gamma_{min}$ can be rigorously proven; (b) the order of magnitude relation $\beta_{min} \sim \gamma_{min}$ takes place in the typical physical situation; (c) β_{min} and γ_{min} are practically equivalent from viewpoint of finite-size scaling philosophy and relations

$$\frac{1}{\gamma_{min}L} = F\left(\frac{L}{\xi}\right) \quad \text{and} \quad \frac{1}{\beta_{min}L} = F\left(\frac{L}{\xi}\right) \quad (14)$$

can be postulated on the same level of rigorousness.

If the correlation length ξ_{1D} is estimated as $1/\beta_{min}$, then the behavior of $g(L)$ determined by Eq. 9 has a form presented in Fig. 1,b; one can see an essential difference from a typical scaling situation (Fig. 1,a). Since there is no growth of $g(L)$ for large L , the state with long-range order (i.e. metallic phase) is absent, in accordance with [8]. Exponential localization takes place for $W > W_c$, while the behavior specific for a critical point is realized in all range $W < W_c$. The latter situation corresponds to localization with the divergent correlation length and probably should be interpreted as power law localization. The transition at $W = W_c$ is of the Kosterlitz-Thouless type and should not be mixed with the usual Anderson transition. This result can explain the observable 2D metal-insulator transition [9] and does not imply a serious revision in the weak localization region.

Violation of scaling relation (12) is clear in Fig. 1,b since $g(L)$ is not constant for $W = W_c$. It is still more evident for $W < W_c$, when different curves have different constant limits for $L \rightarrow \infty$ and surely cannot be made coinciding by a scale transformation.

One may suspect that these results are related with our estimation ξ_{1D} as $1/\beta_{min}$, since the quantities β_{min} and γ_{min} can be essentially different. In fact, substitution of γ_{min} for β_{min} does not lead to qualitative changes in the presented picture and inequality $\beta_{min} > 2\gamma_{min}$ is sufficient for the most responsible statements, i.e. existence of phase transition and violation of scaling [3].

One can see that interpretation in terms of finite size scaling leads to unambiguous conclusion on existence of the 2D phase transition. This statement does not contradict to numerical results, if the raw data are considered [3]. The opposite conclusion made by numerical researchers is based on interpretation in terms of one-parameter scaling, which is inadmissible here. Recent numerical results clearly demonstrate (see Fig. 37 in [7]) that possibility of the 2D phase transition cannot be rejected on numerical grounds.

However, we do not consider the conventional variant of finite size scaling as indisputable. In the general case, relation $\xi_{1D} \sim 1/\gamma_{min}$ can be violated and some effective exponent γ_{eff} should be used instead γ_{min} [3]. If the scaling relation of type (14) is postulated for γ_{eff} , then its dependence on parameters is determined by scaling itself. Analysis shows possibility of two variants: (a) the 2D phase transition is removed; (b) the 2D transition remains, though behavior of the correlation length becomes different. One can suggest that both possibilities are realized in different models.

The described approach can be generalized to higher dimensionality [10, 11]. Unfortunately, the analytical results for the critical disorder appear to be in essential contradiction with corresponding numerical results. The interpretations of this fact can be different, but in any case it is related with crude violation of one-parameter scaling [11].

3. Approach by Kuzovkov et al [1]

Let us discuss the difference of approach presented in Sec.2 from one suggested by Kuzovkov et al [1, 10]. The initial system of equations (4) and its higher dimensional analogue (Eq.5 in [11]) coincide with those used in [1, 10]. However, the quantity $z_{mm'}(n)$ was not introduced in [1, 10] and its role was played by $y_{m'm}(n)$. As a result, the system of equations had no complete difference form and could not be solved in the natural manner with evaluation of full spectrum of exponents β_s . Instead, the authors of [1, 10] used the Z -transform and introduced the so called filter function $H(z)$. The latter, as was demonstrated on simple examples, has the poles corresponding to eigenvalues of the transfer matrix, and, in principle, this approach allows to find the full spectrum of generalized Lyapunov exponents. However, some problems arise in the practical realization of this scheme:

(a) The solution was obtained only in the thermodynamic limit $L \rightarrow \infty$, and the problems exist with interpretation of results (see below).

(b) Averaging over translations in the transversal direction was used for simplification of the problem. This procedure is somewhat questionable since it can lead to elimination of some singularities of $H(z)$. Indeed, translational invariance for the solution (7) of Eq.4 and its higher dimensional analog (Eq.10 in [11]) is absent and averaging over translations eliminates all terms with transverse momentum $\mathbf{p} \neq \mathbf{0}$. Analogously, averaging of the squared solution eliminates all terms with $\mathbf{p} \neq \mathbf{0}, \mathbf{G}/2$ where \mathbf{G} is a vector of a reciprocal lattice corresponding to the main diagonal of the Brillouin zone. We can suggest by comparing results (though cannot prove it rigorously) that the filter function $H(z)$ is analogous to ψ_{nm}^2 and only terms with $\mathbf{p} = \mathbf{0}$ and $\mathbf{p} = \mathbf{G}/2$ are essential for it. Fortunately for the authors of [1, 10], a condition $\mathbf{p} = \mathbf{G}/2$ corresponds to the minimal exponent β_{min} for $d = 2$ and $d \geq 4$, so the critical values σ'_0 (corresponding to W_c in [3, 11]) were found correctly for these cases. In the case $d = 3$, the exponent β_{min} does not correspond to $\mathbf{p} = \mathbf{G}/2$ and a zero value for the critical disorder was obtained in [10] for the band center $E = 0$, in a striking contrast with [11].

Attempt to justify this point made in the paper [12] is not convincing, in our opinion. In fact, the results reduce to the statement that averaging procedure does not eliminate the maximum exponent β_{max} , corresponding to $\mathbf{p} = \mathbf{0}$. This statement is surely correct but it has no relation to physics of the problem.

(c) In the general case, function $H(z)$ can have not only poles corresponding to eigenvalues of the transfer matrix but also another singularities, which are physically irrelevant. In our opinion, it is an origin of the second critical point σ_0 obtained in [1, 10] for higher dimensionality; we see no evidence for it in the spectrum of β_s . Correspondingly, we see no

evidence of a special role of dimensionality $d = 6$, which is surely absent in the exact field theory approach [13].

The papers [1, 10] were formulated in the engineer's language (using the concepts of signals, filters etc.), which has no direct relation to the Anderson transition. The authors relate the extended states with a stable filter and the localized states with unstable filter, i.e. introduce their own localization criterion whose correspondence with conventional one was never studied.

The limit $L \rightarrow \infty$ was taken in [1, 10] from the very beginning, and the finite-size scaling approach could not be used for interpretation of results. As a consequence, the Anderson transition was interpreted as being of the first order. In our opinion, the authors had no serious grounds for such a statement¹ and it was made simply as the only way to make contradiction with all available information not so striking.

Nevertheless, possibility of the first order transition is in conflict with the old ideas by Mott [14]: according to the Poincare theorem, a small change in the energy or the disorder strength induces small changes of the wave functions, and hence the state of the system changes continuously². It is essential, that the Poincare theorem is valid only for a finite system and allows existence of the mobility edge, if the localization radius has divergency in it. Existence of the divergent length is not characteristic for the first order transition.

The possibility of power-law localization was also mentioned in [1, 10], but in the context, which is completely different from that in [3, 11].

4. Approach by Markos et al [2]

Markos et al [2] do not follow the natural procedure of Sec. 2, but advance more complicated approach. They rewrite Eq. 1 using the transfer matrix and construct the direct product of two such matrices. After averaging, they arrive to a linear system of equations determined by a matrix

$$T = \begin{pmatrix} W^2 1 \otimes 1 + D_0 \otimes D_0 & -D_0 \otimes 1 & -1 \otimes D_0 & 1 \otimes 1 \\ D_0 \otimes 1 & 0 & -1 \otimes 1 & 0 \\ 1 \otimes D_0 & -1 \otimes 1 & 0 & 0 \\ 1 \otimes 1 & 0 & 0 & 0 \end{pmatrix}, \quad (15)$$

where $D_0 = E - H_0$ and H_0 is the Hamiltonian of the n -th slice for a pure system, 1 is the unit matrix of the size $L \times L$. In principle, this approach is equivalent to that of Sec. 2 but appears to be practically untractable due to sophisticated matrix constructions. The eigenvalues $\lambda = \exp(iq)$ of the matrix (15) are declared to be determined by equation

$$2 \cos 2q - 2\kappa_i \kappa_j \cos q + (\kappa_i^2 + \kappa_j^2 - 2) = 2W^2 i \sin q \quad (16)$$

¹Two branches of the filter function $H(z)$ were interpreted as two different phases existing simultaneously; such interpretation looks rather vague.

²It does not exclude that certain quantities can display a jump-like behavior; the conductivity, according to Mott, belongs to such quantities.

where $\kappa_i = E - 2 \cos p_i$ and p_i are allowed values of the transverse momentum p . Equation (16) surely not coincides with the corresponding Eq. 9 in Sec. 2. The main difference is the absence of functions with the argument qL , which inevitably arise due to the boundary conditions and can be absent only in trivial cases. One can suspect, that this difference is related with improper treatment of the disorder term (like $W^2 1 \times 1$ in (15)), without which the problem is trivial. This term is local in $m - m'$ (see (4)) and not diagonal in the momentum representation, which is seemingly used in (16). It looks likely that the local nature of this term was neglected and it was replaced by a suitable constant. We can try such thing for the system (4), replacing $\delta_{m,m'}$ by unity. Then (4) is solved trivially

$$x_{m,m'}(n) = x e^{ipm+ip'm'} e^{\beta n}, \quad y_{m,m'}(n) = y e^{ipm+ip'm'} e^{\beta n}, \quad z_{m,m'}(n) = z e^{ipm+ip'm'} e^{\beta n}, \quad (17)$$

where allowed values for p and p' ($2\pi s/L$, $s = 0, 1, \dots, L-1$) are determined by the boundary conditions and the quantities x, y, z satisfy the equation

$$\begin{pmatrix} W^2 + 4 \cos p \cos p' - 2 \sinh \beta & 2 \cos p & 2 \cos p' \\ 2 \cos p & e^\beta & 1 \\ 2 \cos p' & 1 & e^\beta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0. \quad (18)$$

The determinant vanishes under condition

$$2 \cosh 2\beta - 2\kappa\kappa' \cosh \beta + (\kappa^2 + \kappa'^2 - 2) = 2W^2 \sinh \beta \quad (19)$$

where $\kappa = -2 \cos p$, $\kappa' = -2 \cos p'$. In the case $E = 0$ (when Eq. 4 holds) this equation is identical to (16), if correspondence $\beta = iq$ is taken into account. We see that, indeed, the disorder term in (15) was treated inadequately and its local nature was neglected. Physically, equation (16) corresponds not to the true Anderson model but to its degenerate version when site energies $V_{n,m}$ are independent of m .

In fact, the error is present already in the matrix (15). To obtain the term with disorder, one needs to produce averaging of the kind

$$\begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \otimes \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} = \begin{pmatrix} V_1 V_1 & 0 & 0 & 0 \\ 0 & V_1 V_2 & 0 & 0 \\ 0 & 0 & V_2 V_1 & 0 \\ 0 & 0 & 0 & V_2 V_2 \end{pmatrix} \longrightarrow W^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (20)$$

and the result cannot be represented as $W^2 1 \otimes 1$. The latter form is valid for $V_1 = V_2$ in Eq. 20, while in the general case it corresponds to the model (1) with $V_{n,m}$ being independent of m , in accordance with the previous analysis. Such model is of no interest and all conclusions made in [2] are irrelevant for the problem under consideration.

5. Conclusion

The following conclusions can be made from our analysis of generalized Lyapunov exponents:

- (a) A spectrum of β_s allows complete analytical investigation.
- (b) Interpretation of results in terms of conventional variant of finite-size scaling leads to unambiguous conclusion on existence of the 2D transition of the Kosterlitz-Thouless type and crude violation of one-parameter scaling in any dimensionality.
- (c) Attempt to restore one-parameter scaling [8] by replacement γ_{min} by γ_{eff} eliminates the 2D transition from roughly half of models.
- (d) Results for the critical disorder obtained by Kuzovkov et al [1, 10] are correct for $d = 2$ and $d \geq 4$ (not for $d = 3$), but their interpretation is not satisfactory.
- (e) Qualitatively different conclusions by Markos et al [2] are based on improper calculation of the matrix product.
- (f) Above conclusions do not contradict numerical results, if the raw data are considered; interpretation of the latter in terms of one-parameter scaling is inadmissible.

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